Math 1500, Exam III, WS2005

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Do all the problems

If you need more space, use the opposite blank page

and say so.

Calculators are neither needed nor allowed.

You may not leave before 7:30 p.m.

Make sure you have 9 pages including the cover page.

Name____

Student Number__________________________________

RSD Instructor____________________Time your RSD meets______

Do not write below this point.

Part A

__________/30 points

Part B

I. __________/10 points

II. __________/20 points

III. __________/20 points

IV. __________/20 points

SUM __________ Grade:___

Part A: (SHORT ANSWERS) Do the following problems. Write the answer in the space provided. Only the answers will be graded; **there is no partial credit.**
(30 points; 3 points each).

1. If $y = \sqrt{1+2x}$, find dy.

   Answer: $\underline{\dfrac{dx}{\sqrt{1+2x}}}$

2. If $y = \tan(5\theta)$, find dy.

   Answer: $\underline{5\sec^2(5\theta)d\theta}$

3. Find the equation of the **vertical asymptote** of $y = \dfrac{x^3}{x+3}$.

   Answer: $\underline{x = -3}$

4. If $f'(c) = 0$, then either f has a local maximum value at c or f has a local minimum value at c.

   Circle the appropriate answer: Always true   [Can be false]

5. Find the equation of the **horizontal asymptote** of $y = \dfrac{5x^4 - 2x + 1}{3x^4 + x^2 + 10}$.

   Answer: $\underline{y = \dfrac{5}{3}}$

6. The point $(1, -2)$ is an **inflection point** of $y = x^3 - 3x^2$.

   Circle the appropriate answer: $\boxed{\text{True}}$ False

7. On what interval is the function $f(x) = x + 2\cos(x)$ increasing?

   Circle the appropriate answer: $\left(0, \frac{\pi}{2}\right)$ $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ $\boxed{\left(0, \frac{\pi}{6}\right)}$

8. Find $\lim_{x \to \infty} 2x \sin\left(\frac{1}{x}\right)$.

   Answer: $\underline{2}$

9. The function $f(x) = \frac{x^2}{2} + \frac{1}{x^2}$ is concave up on $(-\infty, 0)$.

   Circle the appropriate answer: $\boxed{\text{True}}$ False

10. Find $\lim_{x \to -\infty} \frac{\sqrt{x^2 + 2x}}{1 - 2x}$.

    Answer: $\underline{\frac{1}{2}}$

Part B: For the following problems give a complete solution. Partial credit is possible and you must **SHOW ALL YOUR WORK.**

I) Let $f(x) = \dfrac{x^3}{3} - x^2 + 1$; $-\infty < x < \infty$.

(a) (4 points) Find the intervals on which f is increasing or decreasing.

Answer:

$f'(x) = x^2 - 2x$

$x = 0$ and $x = 2$ are critical numbers

x	$-\infty$		0		2		∞
f'		$+$		$-$		$+$	
f	$-\infty$	$\nearrow$	1	$\searrow$	$-\frac{1}{3}$	$\nearrow$	∞

f is increasing on $(-\infty, 0) \cup (2, \infty)$

f is decreasing on $(0, 2)$

(b) (2 points) Find the local maximum and minimum values of f.

Answer: f has a local maximum value of 1 at $x = 0$

f has a local minimum value of $\dfrac{8}{3} - 4 + 1 = -\dfrac{1}{3}$ at $x = 2$

(c) (4 points) Find the intervals of concavity and inflection point(s).

Answer: $f''(x) = 2x - 2$

x	$-\infty$		1		∞
f''		$-$	0	$+$	
f		$\frown$	$\frac{1}{3}$	$\smile$	

f is concave up on $(1, \infty)$

f is concave down on $(-\infty, 1)$

f has an inflection point at $\left(1, \dfrac{1}{3}\right)$

II) (a) (10 points) Let $f(x) = (1-x)^{\frac{2}{3}}$; $0 \leq x \leq 9$. Find the absolute maximum value and the absolute minimum value of f on $[0, 9]$.

Answer:

$$f'(x) = -\frac{2}{3}(1-x)^{-\frac{1}{3}} = -\frac{2}{3}\frac{1}{(1-x)^{\frac{1}{3}}}$$

$x = 1$ is a critical number of f.

$$f(0) = 1$$
$$f(1) = 0$$
$$f(9) = (-8)^{\frac{2}{3}} = 4$$

f has an absolute minimum value of 0 at $x = 1$
f has an absolute maximum value of 4 at $x = 9$

(b) (10 points) If $y^2 - x^2 = 1$, show that $\dfrac{d^2y}{dx^2} = \dfrac{1}{y^3}$.

Answer:

$$\frac{d}{dx}(y^2 - x^2) = \frac{d}{dx}(1) \Leftrightarrow$$
$$2yy' - 2x = 0 \Rightarrow y' = \frac{x}{y}$$
$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{x}{y}\right) \Leftrightarrow$$
$$y'' = \frac{y - x \cdot y'}{y^2} = \frac{y - \frac{x^2}{y}}{y^2}$$
$$y'' = \frac{y^2 - x^2}{y^3} = \frac{1}{y^3}$$

III) (a) (5 points) State the Mean Value Theorem.

Answer: Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b), then there exists number $a < c < b$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

or

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

(b) (5 points) Show that for any real numbers a, b

$$|\sin(b) - \sin(a)| \leq |b - a|.$$

Answer: Let $f(x) = \sin(x)$, then $f'(x) = \cos(x)$. By the MVT, for any real numbers $a < b$, there exists number $a < c < b$ such that

$$\sin(b) - \sin(a) = \cos(c)(b - a) \Rightarrow$$
$$|\sin(b) - \sin(a)| = |\cos(c)||b - a|$$

Since $|\cos(c)| \leq 1$, we have

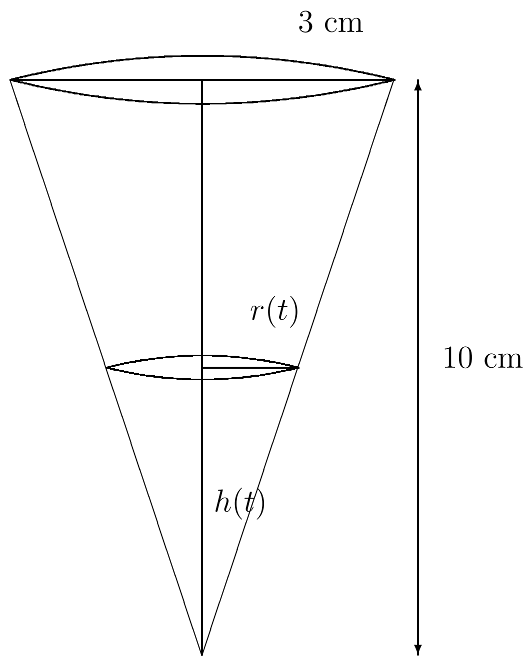
$$|\sin(b) - \sin(a)| = |\cos(c)||b - a| \leq |b - a|.$$

(c) (10 points) Find the linear approximation of the function $f(x) = \sqrt[3]{x}$ at $a = 8$ and use it to approximate $\sqrt[3]{7.99}$.

Answer: Let $f(x) = x^{\frac{1}{3}}$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$$
\begin{aligned}
f(8) = 2 \text{ and } f'(8) &= \frac{1}{12} \\
L(x) &= f(8) + f'(8)(x-8) \\
L(x) &= 2 + \frac{1}{12}(x-8) \\
\sqrt[3]{7.99} &\simeq L(7.99) = 2 + \frac{1}{12}(7.99-8) \\
\sqrt[3]{7.99} &\simeq 2 - \frac{1}{1200}
\end{aligned}
$$

IV) (a) (10 points) A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?



(Tip: Volume$= V = \frac{\pi}{3} r^2 h$**)**

Answer:

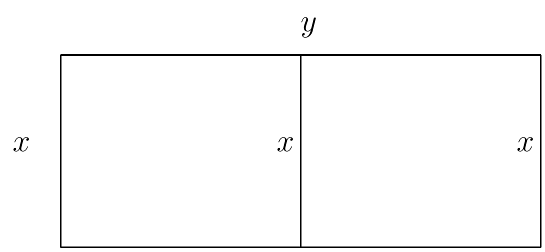
$$\frac{r}{h} = \frac{3}{10} \Rightarrow$$
$$r = \frac{3}{10} h$$
$$V = \frac{\pi}{3} \frac{9}{100} h^2 \cdot h$$
$$V = \frac{9\pi}{300} h^3$$
$$V' = \frac{9\pi}{300} \cdot 3h^2 h' = \frac{9\pi}{100} h^2 h^1$$

when $h = 5$cm and since $V' = 2\text{cm}^3/\text{s}$

$$2 = \frac{9\pi}{100} \cdot 25 h' = \frac{9\pi}{4} h' \Rightarrow$$
$$h' = \frac{8}{9\pi} \text{ so}$$

the water level is rising at the rate of $\frac{8}{9\pi}$cm/s

(b) (10 points) A 600 m^2 **rectangular** garden spot is to be enclosed by a fence and divided into **two equal** parts by fences parallel to one of the sides. What dimensions of the outer rectangle will require the **smallest total length of fence**?



Answer:

$$xy = 600 \Rightarrow y = \frac{600}{x}$$
$$\text{Length } = L = 3x + 2y \Rightarrow$$
$$L(x) = 3x + \frac{1200}{x}$$
$$L'(x) = 3 - \frac{1200}{x^2} = 0 \Leftrightarrow \frac{3x^2 - 1200}{x^2} = 0 \Leftrightarrow$$
$$x^2 = 400, \text{ hence } x = 20\text{m and } y = 30\text{m}$$

Check: $L''(x) = \dfrac{2400}{x^3} > 0$ for all $x > 0$.
$L(x)$ is concave up on $(0, \infty)$.
OR

x	0	20	∞
$L'(x)$	$-$	0 $+$	
$L(x)$	$\searrow$	$\nearrow$	$L(x) = 3x$
	∞	min	

$L(x)$ is minimum when $x = 20$m