Math 1500, Exam I, WS2005

Paulette Saab

Do all the problems

If you need more space, use the opposite blank page
and say so.

Calculators are neither needed nor allowed.

You may not leave before 7:30 p.m.

Name____

Student Number__________________________________

RSD Instructor_________________________ Time your RSD meets______

Do not write below this point.

Part A

__________/30 points

Part B

I. __________/10 points

II. __________/20 points

III. __________/20 points

IV. __________/20 points

SUM __________ Grade:___

Part A: (SHORT ANSWERS) Do the following problems. Write the answer in the space provided. Only the answers will be graded; **there is no partial credit.**
(30 points; 3 points each).

1. What is the domain of the function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$?

   Answer: $-1 < x < 1$ or $(-1, 1)$

2. Compute $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

   Answer: 6

3. If a ball is thrown into the air with a velocity of 20 ft/sec, its height in feet after t seconds is given by $s(t) = 20t - 16t^2$. Find the **average velocity** for the time period $[0, 1]$.

   Answer: 4 ft/sec

4. Express the **area** A of a **square** as a function of its **perimeter** P.

   Answer: $A = \dfrac{P^2}{16}$

5. Compute $\lim_{x \to 5^+} \dfrac{x - 6}{5 - x}$.

   Answer: $+\infty$

6. If $f(2) > 0$ and $f(3) < 0$, then there exists a number c between 2 and 3 such that $f(c) = 0$.

   Circle the appropriate answer: Always true $\qquad$ Can be false

7. Compute $\lim_{x \to 2^-} \dfrac{x-2}{|x-2|}$.

   Answer: -1

8. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$.

   Circle the appropriate answer: Always true $\qquad$ Can be false

9. Let f be a function such that $\lim_{x \to 0} f(x) = 3$. Then there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $|f(x) - 3| < \dfrac{1}{2}$.

   Circle the appropriate answer: True $\qquad$ False

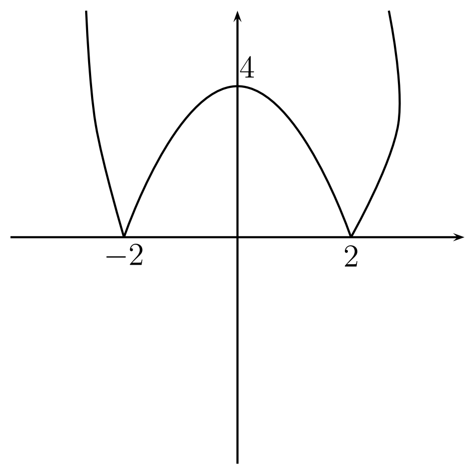
10. If $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$, then $f(x)$ is continuous at $x = \pi$.

    Circle the appropriate answer: True $\qquad$ False

Part B: For the following problems give a complete solution. Partial credit is possible and you must **SHOW ALL YOUR WORK.**

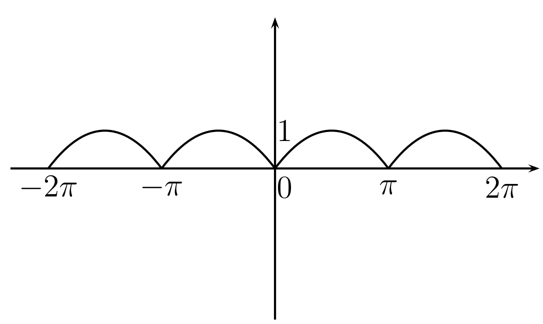
I) (a) (5 points) Sketch the graph of the function $y = |x^2 - 4|$. **Make sure to label the x-intercept, as well as the y-intercept.**

Answer:



(b) (5 points) Sketch the graph of the function $y = |\sin(x)|$, $-2\pi \le x \le 2\pi$. **Make sure to label the x-intercepts, as well as the y-intercept**.

Answer:



II) (a) (10 points) If $f(x) = 1 + \sqrt{x-4}$ and $g(x) = 3 + x^2$, find the composition functions $f \circ g$ and $g \circ f$, and **make sure to specify their domains**.

Answer:

$$\begin{aligned} f \circ g = f(g(x)) &= 1 + \sqrt{3 + x^2 - 4} = 1 + \sqrt{x^2 - 1}; \quad x \geq 1 \text{ or } x \leq -1. \\ g \circ f(x) = g(f(x)) &= 3 + (1 + \sqrt{x-4})^2 = 3 + 1 + 2\sqrt{x-4} + x - 4 \\ &= 2\sqrt{x-4} + x; \quad x \geq 4 \end{aligned}$$

(b) (10 points) Suppose that the inequalities

$$2 - x^4 \leq f(x) \leq 1 + \sqrt{1 + x^2}$$

hold for all real x. Find $\lim_{x \to 0} f(x)$. **Give reasons for your answer**.

Answer: Since $\lim_{x \to 0}(2 - x^4) = 2$, and $\lim_{x \to 0}(+\sqrt{1 + x^2}) = 2$, by the Squeeze Theorem

$$\lim_{x \to 0} f(x) = 2.$$

III) (a) (10 points) Compute $\lim_{t\to 2} \frac{\frac{1}{t}-\frac{1}{2}}{t-2}$.

Answer:

$$\begin{aligned}\lim_{t\to 2} \frac{\frac{1}{t}-\frac{1}{2}}{t-2} &= \lim_{t\to 2} \frac{\frac{2-t}{2t}}{t-2} \\ &= \lim_{t\to 2} \frac{-1(t-2)}{2t(t-2)} \\ &= \lim_{t\to 2} \frac{-1}{2t} = -\frac{1}{4}\end{aligned}$$

(b) (10 points) Let $g(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x > 2 \\ \cos(x-2)+3 & \text{if } x \leq 2 \end{cases}$.

(1) Find $\lim_{x\to 2^+} g(x)$.

Answer: $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \dfrac{x^2-x-2}{x-2} = \lim_{x\to 2^+} \dfrac{(x-2)(x+1)}{x-2} = 3.$

(2) Find $\lim_{x\to 2^-} g(x)$.

Answer: $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \cos(x-2)+3 = 1+3 = 4.$

(3) Is the function g continuous at $x = 2$? **Explain**.

Answer: Since $\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$, then $\lim_{x\to 2} g(x)$ does not exist. Thus g is not continuous at $x = 2$.

IV) (a) (10 points) Use the definition given in chapter 2 to find the slope of the tangent line to the graph of $f(x) = \sqrt{x+5}$ at the point $(4,3)$, then find an equation of the tangent line there.

Answer:

$$\begin{aligned}
m_{\tan} &= \lim_{x\to 4} \frac{f(x)-f(4)}{x-4} \\
&= \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \\
&= \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} \\
&= \lim_{x\to 4} \frac{x+5-9}{(x-4)(\sqrt{x+5}+3)} \\
&= \lim_{x\to 4} \frac{(x-4)}{(x-4)(\sqrt{x+5}+3)} = \frac{1}{6}
\end{aligned}$$

or

$$\begin{aligned}
m_{\tan} &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \\
&= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6}
\end{aligned}$$

The equation of the tangent line is $y - 3 = \frac{1}{6}(x-4)$ or $y = \frac{x}{6} + \frac{7}{3}$

(b) (10 points) Use the definition given in chapter 2 to find the slope of the tangent line to the graph of $f(x) = x^3$ at the point (a, a^3), then find the points on the graph of $y = x^3$ where the slope of the tangent line is equal to 3.

Answer:

$$\begin{aligned} m_{\tan} &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^3 - a^3}{x - a} \\ &= \lim_{x \to a} \frac{(x-1)(x^2 + ax + a^3)}{x - a} = 3a^2 \end{aligned}$$

or

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h \to 0} (3a^2 + 3ah + h^2) = 3a^2 \end{aligned}$$

$$3a^2 = 3 \Leftrightarrow a^2 = 1 \Leftrightarrow a = \pm 1$$

The points on the graph of $y = x^3$ whose slope is equal 3 are:

$$(1, 1) \text{ and } (-1, -1).$$