Sample Questions Exam IV Solution, FS2009

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Calculators are neither needed nor allowed.

Part A: (SHORT ANSWER QUESTIONS) Do the following problems. Write the answer in the space provided. Only the answers will be graded; **there is no partial credit.**

1. Evaluate $\displaystyle\int_1^4 \frac{dt}{2\sqrt{t}}$.

   $$\int_1^4 \frac{dt}{2\sqrt{t}} = \left[\sqrt{x}\,\right]_1^4 = \sqrt{4} - \sqrt{1} = 2 - 1 = 1.$$

2. Evaluate $\displaystyle\int_0^1 (y^9 - 2y^5)dy$.

   $$\int_0^1 (y^9 - 2y^5)dy = \left[\frac{y^{10}}{10} - 2\frac{y^6}{6}\right]_0^1 = \frac{1}{10} - \frac{1}{3}.$$

3. Evaluate $\displaystyle\int 3\cos(3\theta\ )d\theta$.

   $$\int 3\cos(3\theta\ )d\theta = \sin(3\theta) + C.$$

4. Evaluate $\displaystyle\int \tan^3(t)\sec^2(t)\ dt$.

   $$\int \tan^3(t)\sec^2(t)\ dt = \frac{\tan^4(t)}{4} + C.$$

5. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4(\theta)\cos(\theta)d\theta$.

   $$\int_0^{\frac{\pi}{2}} \sin^4(\theta)\cos(\theta)d\theta = \left[\frac{\sin^5(t)}{5}\right]_0^{\frac{\pi}{2}} = \frac{1}{5}.$$

6. Evaluate $\displaystyle\int_{-1}^1 \frac{\sin x}{(1+x^6)^2}dx$.

   Since the function $f(x) = \dfrac{\sin x}{(1+x^6)^2}$ is odd, $\displaystyle\int_{-1}^1 \frac{\sin x}{(1+x^6)^2}dx = 0$

7. If $F(x) = \displaystyle\int_1^x \frac{1}{\sqrt{1+t^4}}dt$, find $F'(x)$.

   $$F'(x) = \frac{1}{\sqrt{1+x^4}}.$$

8. If $g(x) = \int_3^x \cos(1+t^3)\,dt$, find $g'(x)$.

   $g'(x) = \cos(1+x^3)$.

9. $\int_{-1}^{1} \left(x^5 - 6x^9 + \frac{\sin(x)}{(1+x^4)^2}\right) dx = 0.$

   Circle the appropriate answer [✓ **True**] **False**

   Because $f(x) = \left(x^5 - 6x^9 + \frac{\sin(x)}{(1+x^4)^2}\right)$ is odd.

10. $\int_0^2 (x - x^3)dx$ represents the area under the curve $y = x - x^3$ from $x = 0$ to $x = 2$.

    Circle the appropriate answer **True** [✓ **False**]

11. If $F(x) = \int_1^{4x} \sqrt{1+\sin t}\,dt$, find $F'(x)$.
    By the Chain Rule and the Fundamental Theorem of Calculus $F'(x) = \sqrt{1+\sin(4x)}\,(4)$.

12. If f is continuous on $[a, b]$, then

    $$\int_a^b f(x)dx = \text{ area under the graph of } y = f(x) \text{ from } x = a \text{ to } x = b.$$

    Circle the appropriate answer Always True [✓ Can be False]
    It is true if $f(x) \geq 0$ for all x in $[a, b]$.

13. $\int_{-1}^{1} \frac{1}{x^2} dx = -2.$

    Circle the appropriate answer **True** [✓ **False**]
    $\int_{-1}^{1} \frac{1}{x^2} dx$ does not exist.

14. Evaluate $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$.

    $$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \text{area under the upper-half circle of radius } a = \frac{\pi a^2}{2}$$

15. Evaluate $\int_{-1}^{1} \frac{\sin x}{(1+x^6)^2} dx$.

    $$\int_{-1}^{1} \frac{\sin x}{(1+x^6)^2} dx = 0 \text{ since } f(x) = \frac{\sin x}{(1+x^6)^2} \text{ is odd.}$$

16. Evaluate $\int_0^3 \sqrt{9-x^2}\, dx$.

$$\int_0^3 \sqrt{9-x^2}\, dx = \text{area under the quarter of a circle of radius } 3 = \frac{\pi 3^2}{4}.$$

17. If $\int_2^8 f(x)dx = 7$ and $\int_2^5 f(x)dx = 3$, find $\int_5^8 f(x)dx$.

$$\int_5^8 f(x)dx = \int_5^2 f(x)dx + \int_2^8 f(x)dx = -\int_2^5 f(x)dx + \int_2^8 f(x)dx = 7 - 3 = 4.$$

18. If f is continuous on $[a,b]$, then $\int_a^b xf(x)dx = x\int_a^b f(x)dx$.

Circle the appropriate answer: **True** [✓ **False**]

19. Evaluate $\int \sin^6(t)\cos(t)\, dt$.

$$\int \sin^6(t)\cos(t)\, dt = \frac{\sin^7(t)}{7} + C.$$

20. Evaluate $\int \tan^4(3\theta)\sec^2(3\theta)d\theta$.

$$\int \tan^4(3\theta)\sec^2(3\theta)d\theta = \frac{1}{3}\frac{\tan^5(3\theta)}{5} + C = \frac{\tan^5(3\theta)}{15} + C.$$

21. Evaluate $\int_{-1}^1 x\cos(x^3)dx$.

$\int_{-1}^1 x\cos(x^3)dx = 0$ since $f(x) = x\cos(x^3)$ is odd.

22. $\int_{-5}^5 (ax^2 + bx + c)\, dx = 2\int_0^5 (ax^2 + c)\, dx$

Circle the appropriate answer [✓ **True**] **False**

Because $f(x) = ax^2$ and $g(x) = c$ are even functions, and $h(x) = bx$ is odd, $\int_{-5}^5 ax^2\, dx = 2\int_0^5 ax^2\, dx$ and $\int_{-5}^5 c\, dx = 2\int_0^5 c\, dx$, while $\int_{-5}^5 bx\, dx = 0$.
Thus $\int_{-5}^5 (ax^2 + bx + c)\, dx = \int_{-5}^5 ax^2\, dx + \int_{-5}^5 bx\, dx + \int_{-5}^5 c\, dx = 2\int_0^5 (ax^2 + c)\, dx$

Part B: For the following problems give a complete solution. Partial credit is possible and you must **SHOW ALL YOUR WORK.**

I) (a) Evaluate $\int (t^2+1)^3\, dt$.

$\int (t^2+1)^3\, dt = \int t^6 + 3t^4 + 3t^2 + 1\, dt =$

$\frac{t^7}{7} + 3\frac{t^5}{5} + t^3 + t + C.$

(b) (Evaluate $\int_0^2 \frac{2x}{(1+x^2)^2} dx$.

Let $u = 1 + x^2$, then $du = 2xdx$, and

$\int_0^2 \frac{2x}{(1+x^2)^2} dx = \int_1^5 u^{-2}\, du = \left[-u^{-1}\right]_1^5 = \left[-\frac{1}{u}\right]_1^5 = -\frac{1}{5} + 1 = \frac{4}{5}.$

(c) Evaluate $\int_4^9 (\sqrt{x}-1)^3 \frac{dx}{\sqrt{x}}$.

Let $u = \sqrt{x} - 1$, then $du = \frac{1}{2\sqrt{x}} dx$, and $\frac{dx}{\sqrt{x}} = 2du$. It follows that

$\int_4^9 (\sqrt{x}-1)^3 \frac{dx}{\sqrt{x}} = 2\int_1^2 u^3 du = \left[2\frac{u^4}{4}\right]_1^2 = 2 \times 4 - 2 \times \frac{1}{4} = \frac{15}{2}.$

(d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{5+4\sin\theta}} d\theta$.

Let $u = 5 + 4\sin\theta$, then $du = 4\cos(\theta) d\theta$, and $\cos\theta\ d\theta = \frac{du}{4}$. It follows that

$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{5+4\sin\theta}} d\theta = \frac{1}{4}\int_5^9 u^{-\frac{1}{2}} du = \frac{1}{4}\left[2\sqrt{u}\right]_5^9 = \frac{1}{2}\left(3 - \sqrt{5}\right).$

II) (a) Find the area of the region bounded by $y = x^3$ and $y = x$.

The curves intersect when $x^3 = x \iff x = -1, x = 0$ or $x = 1$.
$x^3 - x \geq 0 \iff -1 \leq x \leq 0$ and $x^3 - x \leq 0 \iff 0 \leq x \leq 1$, thus
Area $= \displaystyle\int_{-1}^{0} x^3 - x\, dx + \int_0^1 x - x^3\, dx = \frac{1}{2}$.
or by symmetry,
Area $= 2\displaystyle\int_0^1 x - x^3\, dx = \frac{1}{2}$.

(b) Find the area bounded by $x = y^2$ and $x = 3 - 2y$.

The curves intersect when $y^2 = 3 - 2y \iff y^2 + 2y - 3 = 0 \iff y = -3, \quad$ or $y = 1$.
$y^2 - (3 - 2y) = y^2 + 2y - 3 \geq 0 \iff y \leq -3$ or $y \geq 1$, while
$y^2 - (3 - 2y) = y^2 + 2y - 3 \leq 0 \iff -3 \leq y \leq 1$, thus
Area $= \displaystyle\int_{-3}^{1} 3 - 2y - y^2\, dy = \left[3y - y^2 - \frac{y^3}{3}\right]_{-3}^{1} = \frac{32}{3}$.

(c) Find the area bounded by $y = \sin(x)$, $y = 0$, $x = 0$ and $x = 2\pi$.

Area $= \displaystyle\int_0^{2\pi} |\sin(x)|\, dx = \int_0^{\pi} \sin(x)\, dx + \int_{\pi}^{2\pi} -\sin(x)\, dx =$
$[-\cos(x)]_0^{\pi} + [\cos(x)]_{\pi}^{2\pi} = 1 + 1 + 1 + 1 = 4$.

(d) Find the area bounded by $y = \sin(2x)$, $y = \cos(2x)$, $x = 0$, and $x = \pi$.

The curves intersect when $\sin(2x) = \cos(2x) \iff 2x = \dfrac{\pi}{4}$ or $2x = \pi + \dfrac{\pi}{4}$.
Thus $\sin(2x) = \cos(2x) \iff x = \dfrac{\pi}{8}$ or $x = \dfrac{\pi}{2} + \dfrac{\pi}{8} = \dfrac{5\pi}{8}$. It follows that
Area $= \displaystyle\int_0^{\frac{\pi}{8}} \cos(2x) - \sin(2x)\, dx + \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \sin(2x) - \cos(2x)\, dx + \int_{\frac{5\pi}{8}}^{\pi} \cos(2x) - \sin(2x)\, dx$.
Area$= \dfrac{1}{2}\left[\sin(2x) + \cos(2x)\right]_0^{\frac{\pi}{8}} + \dfrac{1}{2}\left[-\cos(2x) - \sin(2x)\right]_{\frac{\pi}{8}}^{\frac{5\pi}{8}} + \dfrac{1}{2}\left[\sin(2x) + \cos(2x)\right]_{\frac{5\pi}{8}}^{\pi}$.
Area $= 4\sqrt{2}$.

III) (a) If $f(x) = \sqrt{x}$; $1 \leq x \leq 9$, find the **Riemann sum** with $\mathbf{n = 4}$, taking the sample points to be the **midpoints**.

$\Delta x = \dfrac{9-1}{4} = 2.$
The sample points are $x_1^* = 2$, $x_2^* = 4$, $x_3^* = 6$, and $x_4^* = 8$. Then
$\mathcal{R}_4 = 2\left(\sqrt{2} + \sqrt{4} + \sqrt{6} + \sqrt{8}\right)$

(b) If $f(x) = \dfrac{1}{x}$; $0 \leq x \leq 10$, find the **Riemann sum** with $\mathbf{n = 5}$, taking the sample points to be the **right endpoints**.

$\Delta x = \dfrac{10-0}{5} = 2.$
The sample points are $x_1^* = 2$, $x_2^* = 4$, $x_3^* = 6$, $x_4^* = 8$, and $x_5^* = 10$, Then
$\mathcal{R}_4 = 2\left(\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{6} + \dfrac{1}{8} + \dfrac{1}{10}\right).$

(c) Write the following limit as a definite integral and compute its value

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right).$$

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = \int_0^1 \sqrt{x}\, dx =$$
$$\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}.$$

(d) Write the following limit as a definite integral and compute its value

$$\lim_{n\to\infty} \frac{\pi}{n}\left(\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n\pi}{n}\right).$$

$$\lim_{n\to\infty} \frac{\pi}{n}\left(\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n\pi}{n}\right) = \int_0^\pi \cos(x)\, dx =$$
$$[\sin(x)]_0^\pi = 0.$$

IV) Let $\mathcal{R}$ be the region bounded by $\mathbf{y} = \sin \mathbf{x}$, $\mathbf{y} = \mathbf{0}$, $\mathbf{x} = \mathbf{0}$ **and** $\mathbf{x} = \pi$.

(a) Set up the integral **(DO NOT EVALUATE IT)** that would give you the volume of the solid obtained by rotating the region $\mathcal{R}$ **about the** y**-axis.**

Using cylindrical shells, we get
Volume $= \displaystyle\int_0^\pi 2\pi x \sin(x)\, dx.$

(b) Set up the integral **(DO NOT EVALUATE IT)** that would give you the volume of the solid obtained by rotating the region $\mathcal{R}$ **about the** x**-axis.**

Using Cross sections that are disks, we get
Volume $= \displaystyle\int_0^\pi \pi \sin^2(x)\, dx.$

(c) Set up the integral **(DO NOT EVALUATE IT)** that would give the volume of the solid obtained by rotating the region $\mathcal{R}$ **about** $\mathbf{x} = \pi$**.**

Using cylindrical shells, we get
Volume $= \displaystyle\int_0^\pi 2\pi(\pi - x)\sin(x)\, dx.$

(d) Set up the integral **(DO NOT EVALUATE IT)** that would give you the volume of the solid obtained by rotating the region $\mathcal{R}$ **about** $\mathbf{y} = \mathbf{2}$**.**

Using Cross sections that are washers, we get
Volume $= \displaystyle\int_0^\pi \pi\left(2^2 - (2 - \sin(x))^2\right)\, dx.$

V) (a) Find the **volume** of the solid generated when the region bounded by $y = x^2$, $y = 0$, $x = 0$, and $x = 2$ is revolved about the x-axis.

Using Cross sections that are disks, we get

$$\text{Volume} = \int_0^2 \pi(x^2)^2 \, dx = \int_0^2 \pi x^4 \, dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}.$$

(b) Find the **volume** of the solid generated when the region bounded by $y = x^2$, $y = 0$, $x = 0$, and $x = 2$ is revolved about the y-axis.

Using cylindrical shells, we get

$$\text{Volume} = \int_0^2 2\pi x x^2 \, dx = \int_0^2 2\pi x^3 \, dx = 2\pi \left[\frac{x^4}{4}\right]_0^2 = 8\pi.$$

or

Using Cross sections that are washers, we get

$$\text{Volume} = \int_0^4 \pi(2^2 - (\sqrt{y})^2) \, dy = \int_0^4 \pi(4 - y) \, dy = \pi \left[4y - \frac{y^2}{2}\right]_0^4 = 8\pi.$$

(c) Find the **volume** of the solid generated when the region in the first quadrant, bounded by $y = x^2$, and $y = 1$ is revolved about the x-axis.

Using cross sections that are washers, we get

$$\text{Volume} = \int_0^1 \pi(1^2 - (x^2)^2) \, dx = \int_0^1 \pi(1 - x^4) \, dx = \pi \left[x - \frac{x^5}{5}\right]_0^1 = \frac{4\pi}{5}.$$

(d) Find the **volume** of the solid generated when the region in the first quadrant, bounded by $y = x^2$, and $y = 1$ is revolved about the y-axis.

Using cylindrical shells, we get

$$\text{Volume} = \int_0^1 2\pi x(1 - x) \, dx = \int_0^1 2\pi x(1 - x^2) \, dx =$$

$$\int_0^1 2\pi(x - x^3) \, dx = 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

Or using cross sections that are disks, we get

$$\text{Volume} = \int_0^1 \pi(\sqrt{y})^2 \, dy = \int_0^1 \pi y \, dy = \pi \left[\frac{y^2}{2}\right]_0^1 \, dy = \frac{\pi}{2}.$$

VII) (a) If $F(x) = \displaystyle\int_{2x}^{x^3} \sqrt{1+t^2}\, dt$, find the derivative $F'(x)$ of the function F.

$F'(x) = \sqrt{1+(x^3)^2}\,(3x^2) - \sqrt{1+(2x)^2}\,(2)$. Thus
$F'(x) = 3x^2\,\sqrt{1+x^6} - 2\,\sqrt{1+4x^2}$.

(b) If $F(x) = \displaystyle\int_{x^2}^{\sin(x)} \frac{t}{2+t}\, dt$, find the derivative $F'(x)$ of the function F.

$F'(x) = \dfrac{\sin(x)}{2+\sin(x)}\,\cos(x) - \dfrac{x^2}{2+x^2}\,(2x)$. Thus
$F'(x) = \dfrac{\sin(x)\cos(x)}{2+\sin(x)} - \dfrac{2x^3}{2+x^2}$.

(c) If f is a continuous function such that $\displaystyle\int_0^x f(t)dt = x^2 \sin x + \int_x^0 f(t)\sqrt{t}\, dt$ for all x, find an explicit formula for $f(x)$.

$$\left[\int_0^x f(t)dt = x^2\sin(x) + \int_x^0 f(t)\sqrt{t}\, dt\right] \Longleftrightarrow \left[\int_0^x f(t)dt = x^2\sin(x) - \int_0^x f(t)\sqrt{t}\, dt\right].$$

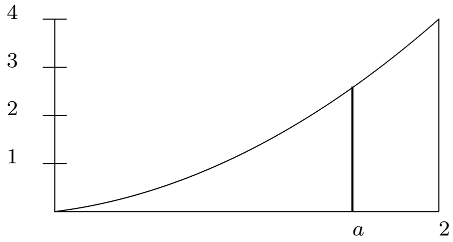
Differentiating with respect to x, we get

$$\frac{d}{dx}\left(\int_0^x f(t)dt\right) = \frac{d}{dx}\left(x^2\sin(x) - \int_0^x f(t)\sqrt{t}\, dt\right) \Longrightarrow$$
$f(x) = x^2\cos(x) + 2x\sin(x) - f(x)\sqrt{x}$.

Solving for $f(x)$, we get

$$f(x) = \frac{x^2\cos(x) + 2x\sin(x)}{1+\sqrt{x}}.$$

VIII (a) (10 points) Find the number a such that the line $x = a$ partitions the region bounded by $y = x^2$, the x-axis, $x = 0$, and $x = 2$, into two regions of **equal** area.



Since the area bounded by $y = x^2$, $x = 0$, and $x = 2$ is twice the area bounded by $y = x^2$, $x = 0$, and $x = a$, we need to solve for a in the following:

$$2\int_0^a x^2\,dx = \int_0^2 x^2\,dx$$

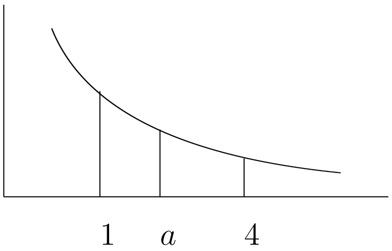
It follows

$$2\left[\frac{x^3}{3}\right]_0^a = \left[\frac{x^3}{3}\right]_0^2.$$

This implies

$$2\frac{a^3}{3} = \frac{8}{3} \Longrightarrow 2a^3 = 8 \Longrightarrow a = \sqrt[3]{4}.$$

(b) Find the number a such that the line $x = a$ partitions the region bounded by $y = \dfrac{1}{x^2}$, the x-axis, $x = 1$, and $x = 4$, into two regions of **equal** area.



Since the area bounded by $y = \dfrac{1}{x^2}$, $x = 1$, and $x = 4$ is twice the area bounded by $y = \dfrac{1}{x^2}$, $x = 1$, and $x = a$, we need to solve for a in the following:

$$2\int_1^a \frac{1}{x^2}\,dx = \int_1^4 \frac{1}{x^2}\,dx$$

It follows

$$2\left[-\frac{1}{x}\right]_1^a = \left[-\frac{1}{x}\right]_1^4.$$

This implies

$$2\left(-\frac{1}{a}+1\right) = -\frac{1}{4}+1 \Longrightarrow -\frac{2}{a} = -\frac{5}{4} \Longrightarrow a = \frac{8}{5}.$$

IX (a) Evaluate $\int_3^4 x\sqrt{x-3}\,dx$. (**Hint:** Let $u = x - 3$.)

Let $u = x - 3$, then $du = dx$ and $x = u + 3$. It follows that

$$\int_3^4 x\sqrt{x-3}\,dx = \int_0^1 (u+3)\sqrt{u}\,du = \int_0^1 (u+3)u^{\frac{1}{2}}\,du =$$

$$\int_0^1 (u^{\frac{3}{2}} + 3u^{\frac{1}{2}})\,du = \left[\frac{2}{5}u^{\frac{5}{2}} + 3 \times \frac{2}{3}u^{\frac{3}{2}}\right]_0^1 = \frac{2}{5} + 2 = \frac{12}{5}.$$

(b) Use the substitution $\mathbf{u} = \mathbf{x} + \mathbf{3}$ to evaluate the indefinite integral

$$\int x\sqrt{x+3}\,dx$$

Let $u = x + 3$, then $du = dx$ and $x = u - 3$. It follows that

$$\int x\sqrt{x+3}\,dx = \int (u-3)\sqrt{u}\,du = \int_0^1 (u-3)u^{\frac{1}{2}}\,du =$$

$$\int (u^{\frac{3}{2}} - 3u^{\frac{1}{2}})\,du = \left[\frac{2}{5}u^{\frac{5}{2}} - 3 \times \frac{2}{3}u^{\frac{3}{2}}\right] + C = \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.$$