Sample Questions Exam III, FS2009

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Calculators are neither needed nor allowed.

Part A: (SHORT ANSWER QUESTIONS) Do the following problems. Write the answer in the space provided. Only the answers will be graded; **there is no partial credit.**

1. If $f'(x) = \sec^2(x)$, and $f(0) = 2$, find $f(x)$.

   $f(x) = \tan(x) + C$, $f(0) = 2 \Longrightarrow C = 2$, thus
   $f(x) = \tan(x) + 2$.

2. If $g'(x) = x\sqrt[3]{x}$, and $g(1) = 1$, find $g(x)$.

   $g'(x) = x^{\frac{4}{3}} \Longrightarrow g(x) = \frac{3}{7}x^{\frac{7}{3}} + C$, $g(1) = 1 \Longrightarrow C = \frac{4}{7}$, thus $g(x) = \frac{3}{7}x^{\frac{7}{3}} + \frac{4}{7}$.

3. The function $f(x) = 5x + \sin(x)$ is concave upward on $(0, \pi)$.

   Circle the appropriate answer True [False]
   Since $f''(x) = -\sin(x) < 0$ on $(0, \pi)$.

4. Find the equation of the horizontal asymptote of $y = \dfrac{2x^3 + 5}{3x^3 - 1}$.

   $y = \dfrac{2}{3}$

5. If f has an absolute minimum value at c, then $f'(c) = 0$.

   Circle the appropriate answer Always True [Can be False]
   $f(x) = |x|$ has an absolute minimum at $x = 0$, but $f'(0)$ does not exist.

6. If $f'(c) = 0$, then f has a local minimum or a local maximum value at c.

Circle the appropriate answer Always True $\boxed{\text{Can be False}}$

If $f(x) = x^3$, $f'(0) = 0$, but f does not have a local min nor a local max at $x = 0$. The derivative $f'(x) = 3x^2$ does not change sign at $x = 0$.

7. Find the equation of the vertical asymptotes of $y = \dfrac{2x}{x^2 - 9}$.

$x = \pm 3$.

8. Find $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{3x^2+1}}$.

$$\lim_{x\to-\infty} \frac{x}{\sqrt{3x^2+1}} = \lim_{x\to-\infty} \frac{x}{|x|\sqrt{3+\frac{1}{x^2}}} = \lim_{x\to-\infty} \frac{x}{(-x)\sqrt{3+\frac{1}{x^2}}} = \lim_{x\to-\infty} -\frac{1}{\sqrt{3+\frac{1}{x^2}}} = -\frac{1}{\sqrt{3}}$$

9. Use Newton's method with $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^3 - 4 = 0$.

$f(x) = x^3 - 4 \Longrightarrow f'(x) = 3x^2$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Longrightarrow x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1-4}{3} = 1 + 1 = 2$$

10. $\displaystyle\lim_{x\to\infty} x \sin\left(\frac{1}{3x}\right)$ is:

Circle the appropriate answer: ∞ 0 $\boxed{\frac{1}{3}}$

Let $t = \dfrac{1}{3x}$, then $x = \dfrac{1}{3t}$, and $x \to \infty \Longleftrightarrow t \to 0^+$

$$\lim_{x\to\infty} x \sin\left(\frac{1}{3x}\right) = \lim_{t\to 0^+} \frac{1}{3t}\sin(t) = \frac{1}{3}\lim_{t\to 0^+} \frac{\sin(t)}{t} = \frac{1}{3}$$

11. The function $f(x) = 5x + \cos(x)$ is increasing on $(-\infty, \infty)$.

Circle the appropriate answer $\boxed{\text{True}}$ False
$f'(x) = 5 - \cos(x) > 0$ for all $-\infty < x < \infty$.

12. If $f''(c) = 0$, then $(c, f(c))$ is an inflection point of the curve $y = f(x)$.

Circle the appropriate answer Always True $\boxed{\text{Can be False}}$
If $f(x) = x^4$, $f'(x) = 4x^3$, $f''(x) = 12x^2$, then $f''(0) = 0$ but f does not have an inflection point at $(0, 0)$ since $f''(x) = 12x^2$ does not change sign at $x = 0$.

13. If f has an absolute minimum value at c, and f is differentiable at c, then $f'(c) = 0$.

Circle the appropriate answer $\boxed{\text{Always True}}$ Can be False

14. Find the equations of the vertical asymptotes of $y = \dfrac{x}{x^2 - 4}$.

$x = \pm 2$

15. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x?

Circle the appropriate answer Yes $\boxed{\text{No}}$
Because by the MVT, there exists $0 < c < 2$, such that

$$f(2) - f(0) = f'(c)(2 - 0) \text{ which implies } 4 - (-1) = f'(c)(2 - 0).$$

Thus there exists $0 < c < 2$, such that

$$5 = 2f'(c) \text{ which implies } 5 \leq 4 \text{ since } f'(c) \leq 2.$$

16. The point $(0, 0)$ is an inflection point of the curve $y = x^4 + 2x^2$.

Circle the appropriate answer True [False]
$y' = 4x^3 + 4x$ and $y'' = 12x^2 + 4 > 0$ for all $-\infty < x < \infty$
The second derivative does not change sign at $x = 0$.

17. Find the most general antiderivative of the function $f(x) = 3x^2 - \dfrac{1}{x^2}$.

$F(x) = x^3 + \dfrac{1}{x} + C$

18. Find $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{x+1}$.

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{x+1} = \lim_{x\to-\infty} \frac{|x|\sqrt{4+\frac{1}{x^2}}}{x(1+\frac{1}{x})} = \lim_{x\to-\infty} \frac{-x\sqrt{4+\frac{1}{x^2}}}{x(1+\frac{1}{x})} = \lim_{x\to-\infty} -\frac{\sqrt{4+\frac{1}{x^2}}}{(1+\frac{1}{x})} = -2$$

19. Find $\displaystyle\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{1-x}$.

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{1-x} = \lim_{x\to\infty} \frac{|x|\sqrt{4+\frac{1}{x^2}}}{x(-1+\frac{1}{x})} = \lim_{x\to\infty} \frac{x\sqrt{4+\frac{1}{x^2}}}{x(-1+\frac{1}{x})} = \lim_{x\to\infty} \frac{\sqrt{4+\frac{1}{x^2}}}{(-1+\frac{1}{x})} = -2$$

20. Find the most general antiderivative of the function $f(x) = 2x + \sqrt{x}$.

Since $f(x) = 2x + x^{\frac{1}{2}}$, the most general antiderivative of f is
$F(x) = x^2 + \dfrac{2}{3}x^{\frac{3}{2}} + C$.

Part B: For the following problems give a complete solution. Partial credit is possible and you must **SHOW ALL YOUR WORK.**

I) (a) Let $f(x) = 4x^3 - 3x^2 - 6x + 1$ for $\infty < x < \infty$.

(a) (4 points) Determine the intervals where f is increasing or decreasing.

$f'(x) = 12x^2 - 6x - 6 = 6(2x+1)(x-1)$

$f'(x) = 0 \iff x = 1 \text{ or } x = -\frac{1}{2}$.

Critical numbers are $-\frac{1}{2}$ and 1.

Moreover, $f'(x) > 0 \iff x < -\frac{1}{2}$ or $x > 1$ and $f'(x) < 0 \iff -\frac{1}{2} < x < 1$.

Thus f is **increasing** on $\left(-\infty, -\frac{1}{2}\right) \bigcup (1, \infty)$, and

f is **decreasing** on $\left(-\frac{1}{2}, 1\right)$.

(b) (2 points) Find the local maximum and minimum values of f.

f has a local max value at $x = -\frac{1}{2}$, $f\left(-\frac{1}{2}\right) = \frac{11}{4}$.

f has a local min value at $x = 1$, $f(1) = -4$.

(c) (4 points) Determine the intervals where the graph of f is concave up or concave down and identify the inflection point(s).

$f''(x) = 24x - 6 = 0 \iff x = \frac{1}{4}$

Moreover, $f''(x) > 0 \iff x > \frac{1}{4}$ while $f''(x) < 0 \iff x < \frac{1}{4}$.

Thus f is **concave up** on $\left(\frac{1}{4}, \infty\right)$ and f is **concave down** on $\left(-\infty, \frac{1}{4}\right)$

f has an **inflection** point at $\left(\frac{1}{4}, -\frac{5}{8}\right)$.

II) (a) Let $f(x) = x^2 + \frac{2}{x}$; $\frac{1}{2} \leq x \leq 2$. Find the absolute maximum value and the absolute minimum value of f on $[\frac{1}{2}, 2]$.

$f'(x) = 2x - \frac{2}{x^2} = 0 \Longleftrightarrow x^3 = 1 \Longleftrightarrow x = 1.$

Thus $x = 1$ is the only critical number of f on $\left[\frac{1}{2}, 2\right]$.

Since $f\left(\frac{1}{2}\right) = \frac{1}{4} + 4, \quad f(1) = 3, \text{ and } f(2) = 5,$

f has an absolute minimum value of 3 at $x = 1$

f has an absolute maximum value of 5 at $x = 2$

(b) Let $f(x) = (x-2)^{\frac{2}{3}}$; $1 \leq x \leq 10$. Find the absolute maximum value and the absolute minimum value of f on $[1, 10]$.

$f'(x) = \frac{2}{3}(x-2)^{-\frac{1}{3}}.$

Thus $x = 2$ is the only critical number of f on $[1, 10]$.

Since $f(1) = 1, \quad f(2) = 0, \text{ and } \quad f(10) = 4,$

f has an absolute minimum value of 0 at $x = 2$

f has an absolute maximum value of 4 at $x = 10$

(c) Compute

$$\lim_{x\to\infty}(\sqrt{9x^2+2x}-3x)$$

$$\lim_{x\to\infty}(\sqrt{9x^2+2x}-3x)=\lim_{x\to\infty}(\sqrt{9x^2+2x}-3x)\frac{(\sqrt{9x^2+2x}+3x)}{(\sqrt{9x^2+2x}+3x)}=$$
$$\lim_{x\to\infty}\frac{9x^2+2x-9x^2}{(\sqrt{9x^2+2x}+3x)}=\lim_{x\to\infty}\frac{2x}{|x|\sqrt{9+\frac{2}{x}}+3x}=$$
$$\lim_{x\to\infty}\frac{2x}{x(\sqrt{9+\frac{2}{x}}+3)}=\lim_{x\to\infty}\frac{2}{(\sqrt{9+\frac{2}{x}}+3)}=\frac{2}{\sqrt{9}+3}=\frac{1}{3}$$

(d) **Use the Squeeze Theorem** to explain why $\lim_{x\to\infty}\frac{\cos^2(x)}{x^2}=0$.

Since $-1\le\cos(x)\le 1$ for all real number x, we have

$$0\le\cos^2(x)\le 1 \text{ for all real number x}$$

It follows that for any real number x

$$0\le\frac{\cos^2(x)}{x^2}\le\frac{1}{x^2} \qquad (*)$$

Since $\lim_{x\to\infty}0=\lim_{x\to\infty}\frac{1}{x^2}=0$,

it follows from (*), and **the Squeeze Theorem** that

$$\lim_{x\to\infty}\frac{\cos^2(x)}{x^2}=0$$

III a) Apply Newton's method to the equation $\boldsymbol{\frac{1}{x} - a = 0}$ to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - ax_n^2.$$

Let $f(x) = \frac{1}{x} - a$, then $f'(x) = -\frac{1}{x^2}$. By Newton's method

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = x_n + x_n - ax_n^2.$

Thus

$\boxed{x_{n+1} = 2x_n - ax_n^2.}$

III b) Apply Newton's method to the equation $\boldsymbol{x^2 - a = 0}$ to derive the following square-root algorithm:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Let $f(x) = x^2 - a$, then $f'(x) = 2x$. By Newton's method

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - x_n^2 + a}{2x_n}.$

It follows that

$$x_{n+1} = \frac{x_n^2 + a}{2x_n}$$

Thus

$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)}$

IV) (a) Show that for any $x > 0$

$$\sqrt{1+x} < 1 + \frac{1}{2}x$$

Let $x > 0$. By the Mean Value Theorem applied to the funtion

$$f(t) = \sqrt{1+t} = (1+t)^{\frac{1}{2}}$$

on the interval $[0, x]$, there exists $0 < c < x$ such that

$$f(x) - f(0) = f'(c)(x - 0)$$

Since $f'(t) = \frac{1}{2}(1+t)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+t}}$, we get

$$\sqrt{1+x} - 1 = \frac{1}{2\sqrt{1+c}}(x) = \frac{1}{\sqrt{1+c}}\frac{1}{2}x.$$

Since $c > 0$, we get $1 + c > 1$, and hence $\sqrt{1+c} > 1$, which implies

$$\frac{1}{\sqrt{1+c}} < 1$$

Thus

$$\sqrt{1+x} - 1 < \frac{1}{2}x \Longleftrightarrow \sqrt{1+x} < 1 + \frac{1}{2}x$$

(b) Show that for any $x > 0$

$$\sqrt{1+x^2} < 1 + x$$

Let $x > 0$. By the Mean Value Theorem applied to the funtion

$$f(t) = \sqrt{1+t^2} = (1+t^2)^{\frac{1}{2}}$$

on the interval $[0, x]$, there exists $0 < c < x$ such that

$$f(x) - f(0) = f'(c)(x - 0)$$

Since $f'(t) = \frac{1}{2}(1+t^2)^{-\frac{1}{2}}(2t) = \frac{t}{\sqrt{1+t^2}}$, we get

$$\sqrt{1+x^2} - 1 = \frac{c}{\sqrt{1+c^2}}(x).$$

Since $c > 0$, and $1 + c^2 > c^2$, we get $\sqrt{1+c^2} > c$, which implies

$$\frac{c}{\sqrt{1+c^2}} < 1$$

Thus

$$\sqrt{1+x^2} - 1 < x \Longleftrightarrow \sqrt{1+x^2} < 1 + x$$

(c) Show that for any real numbers a and b we have

$$|\cos(b) - \cos(a)| \leq |b - a|$$

Let $f(x) = \cos(x)$. Then $f'(x) = -\sin(x)$.
By the Mean Value Theorem, for any real numbers a and b, there exists c between a and b such that

$$f(b) - f(a) = f'(c)(b - a)$$

Thus, there exists c between a and b such that

$$\cos(b) - \cos(a) = -\sin(c)(b - a)$$

Taking the absolute value of both sides, we get

$$|\cos(b) - \cos(a)| = |\sin(c)||b - a|$$

Since for any real number x, $-1 \leq \sin(x) \leq 1 \iff |\sin(x)| \leq 1$, we get

$$|\cos(b) - \cos(a)| = |\sin(c)||b - a| \leq |b - a|$$

(d) Use Rolle's theorem to show that the equation $x^5 + 10x + 3 = 0$ has at most one real root.

Let $f(x) = x^5 + 10x + 3$. Since f is contiuous on $(-\infty, \infty)$ and since

$$f(-1) = -7 < 0 \text{ and } f(0) = 3 > 0,$$

by the **Intermediate Value Theorem**, there exists $-1 < r < 0$ such that $f(r) = 0$. It follows that f has a real root between -1 and 0.
Suppose f had two distinct real roots a and b, with $a < b$. Then

$$f(a) = f(b) = 0$$

Since f is a polynomial then
1) f is continuous on $[a, b]$,
2) f is differentiable on (a, b), and
3) $f(a) = f(b) = 0$

By **Rolle's Theorem**, there exists $a < c < b$ such that

$$f'(c) = 0$$

but $f'(x) = 5x^4 + 10 \neq 0$ for all x real. This leads to a contradiction, and therefore f has only *one* real root.

VI) (a) A car is travelling at 60 mi/h = 88 ft/s, when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance traveled before the car comes to a stop?

We need to solve the initial value problem

$$s''(t) = -22, \ s'(0) = 88 \text{ and } s(0) = 0.$$

$$s''(t) = -22, \text{ and } s'(0) = 88 \implies s'(t) = -22t + 88$$
$$s'(t) = -22t + 88 \text{ and } s(0) = 0 \implies s(t) = -11t^2 + 88t.$$

The car comes to a stop when $s'(t) = 0$, thus

$$s'(t) = -22t + 88 = 0 \implies t = 4s.$$

The distance traveled before the car comes to a stop is

$$s(4) = -11 \times 16 + 88 \times 4 = 176ft.$$

(b) A car braked with a constant deceleration of 16 ft/s^2, producing skid marks measuring 200ft before coming to a stop. How fast was the car travelling when the brakes were first applied?

We need to solve the initial value problem

$$s''(t) = -16, \ s'(0) = v_0 \text{ and } s(0) = 0.$$

$$s''(t) = -16, \text{ and } s'(0) = v_0 \implies s'(t) = -16t + v_0$$
$$s'(t) = -16t + v_0 \text{ and } s(0) = 0 \implies s(t) = -8t^2 + v_0 t.$$

The car comes to a stop when $s'(t) = 0$, thus

$$s'(t) = -16t + v_0 = 0 \implies t = \frac{v_0}{16}s.$$

The distance traveled before the car comes to a stop is

$$s(\frac{v_0}{16}) = -8(\frac{v_0}{16})^2 + v_0(\frac{v_0}{16}) = 200ft \implies$$

$$-\frac{v_0^2}{32} + \frac{v_0^2}{16} = 200 \implies \frac{v_0^2}{32} = 200 \implies$$
$$v_0^2 = 6400 \implies v_0 = 80ft/s.$$

VII) (a) Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

Let (x, y) be an arbitrary point on the ellipse $4x^2+y^2 = 4$. Let $d^2 = (x-1)^2+y^2$ be square of the distance from (x, y) to $(1, 0)$.
Since $y^2 = 4 - 4x^2$, $d^2 = (x - 1)^2 + 4 - 4x^2$.
Thus we need to maximize $f(x) = (x - 1)^2 + 4 - 4x^2,\ -1 \leq x \leq 1$.
$f'(x) = 2(x - 1) - 8x = -6x - 2 = 0 \Longleftrightarrow x = -\dfrac{1}{3}$.
Check
Since $f(-1) = 4,\ f(-\frac{1}{3}) = 5\frac{1}{3}\ f(1) = 0$, it follows that f is maximum when $x = -\dfrac{1}{3}$. Thus we have two points on the ellipse where this maximum is atteined:

$$\left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right) \text{ and } \left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right)$$

(b) The **sum** of two non negative numbers is 40. Find the numbers if **one of the numbers plus four times the square root of the other** is to be as **large as possible**.

Let x be one of the numbers, the other is $y = 40 - x$.
We need to maximize

$$f(x) = x + 4\sqrt{40 - x},\ 0 \leq x \leq 40$$

$$f'(x) = 1 - \frac{2}{\sqrt{40 - x}} = 0 \Longleftrightarrow \sqrt{40 - x} = 2$$

$$40 - x = 4 \Longrightarrow x = 36.$$

Check Since $f(0) = 8\sqrt{10},\ f(36) = 44$ and $f(40) = 40,\ f$ is maximum when $x = 36$. The numbers are $x = 36$ and $y = 4$.

(c) A box with **square base** and **open top** must have a **volume of 32 in^3**. Find the dimensions of the box that **minimize** the amount of material used. (Neglect thickness of material and waste in construction.)

The surface area $S = 4xh + x^2$. Since the volume $V = x^2 h = 32$

we get that $h = \dfrac{32}{x^2}$. Thus we need to minimize

$$S(x) = x^2 + \frac{128}{x}, \quad 0 < x < \infty$$

$S'(x) = 2x - \dfrac{128}{x^2} = 0 \Longleftrightarrow 2x^3 = 128 \Longleftrightarrow x = 4\text{in}.$

Check

A. Since $x > 0$, $S'(x) = \dfrac{2(x^3-64)}{x^3} > 0 \Longleftrightarrow x^3 > 64 \Longleftrightarrow x > 4$ while $S'(x) = \dfrac{2(x^3-64)}{x^3} < 0 \Longleftrightarrow x^3 < 64 \Longleftrightarrow 0 < x < 4.$
By the **First derivative test** S is minimum when $x = 4$in and $h = 2$in.

OR

B. Since $S''(x) = 2 + \dfrac{256}{x^3} > 0$ for all $x > 0$, $S(x)$ is concave up on $(0, \infty)$. Thus S is minimum when $x = 4$in and $h = 2$in, by the **Second derivative test**.

(b) A 150 m^2 **rectangular** garden spot is to be enclosed by a fence and divided into **two equal** parts by fences parallel to one of the sides. What dimensions for the outer rectangle will require the **smallest total length of fence**?

Let x and y be the dimension of the rectangular garden.
The length is $L = 3x + 2y$. Since the area $A = xy = 150$, $y = \dfrac{150}{x}$, thus we need to minimize

$$L(x) = 3x + \frac{300}{x}, \; 0 < x < \infty$$

$L'(x) = 3 - \dfrac{300}{x^2} = 0 \Longrightarrow x^2 = 100 \Longrightarrow x = 10$ m.

Check

A. Since $x > 0$, $L'(x) = \dfrac{3x^2 - 300}{x^2} > 0 \Longleftrightarrow x^2 > 100 \Longleftrightarrow x > 10$ while $L'(x) = \dfrac{3x^2-300}{x^2} < 0 \Longleftrightarrow x^2 < 100 \Longleftrightarrow x < 10$
Thus by the **First derivative test** L is minimum when $x = 10$m and $y = 15$m.

OR

B. Since $L''(x) = \dfrac{600}{x^3} > 0$ for all $x > 0$, $L(x)$ is concave up on $(0, \infty)$, by the **Second derivative test**, L is minimum when $x = 10$m and $y = 15$m.