Sample Questions Exam II, FS2009

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Calculators are neither needed nor allowed.

Part A: (SHORT ANSWER QUESTIONS) Do the following problems. Write the answer in the space provided. Only the answers will be graded; **there is no partial credit.**

1. If $f'(9) = 10$, find $\dfrac{d}{dx}f(x^2)$ when $x = 3$.

   Answer: $\boxed{60}$

2. If $y = \cos^3 x$, find $\dfrac{dy}{dx}$.

   Answer: $-3\cos^2 x \sin x$

3. $\dfrac{d^2y}{dx^2} = (\dfrac{dy}{dx})^2$

   Circle the appropriate answer **True** $\boxed{\textbf{False}}$

4. If f is a positive differentiable function, then $\dfrac{d}{dx}\sqrt{f(x)} =$

   Circle the appropriate answer $\dfrac{1}{2\sqrt{f(x)}}$ $\boxed{\dfrac{f'(x)}{2\sqrt{f(x)}}}$

5. Find $\lim_{\theta \to 0} \dfrac{\sin(5\theta)}{\sin(8\theta)}$.

   Answer: $\dfrac{5}{8}$

6. If $h(x) = \sqrt{x^2 + 16}$, then

$$\lim_{x \to 3} \frac{h(x) - h(3)}{x - 3} =$$

Circle the appropriate answer $\frac{3}{10}$ $\boxed{\frac{3}{5}}$ $\frac{1}{10}$

7. $\frac{d}{d\theta} \tan^2 \theta = \frac{d}{d\theta} \sec^2 \theta$.

Circle the appropriate answer: $\boxed{\text{True}}$ False

8. If $f'(a)$ exists, then $\lim_{x \to a} f(x) = f(a)$.

Circle the appropriate answer $\boxed{\text{Always True}}$ Can be False

9. Find $\lim_{\theta \to 0} \frac{\tan(4\theta)}{7\theta}$.

Answer: $\frac{4}{7}$

10. If $y = x^3 \sin(x)$, find $\frac{dy}{dx}$.

Answer: $x^3 \cos(x) + 3x^2 \sin(x)$

11. If $y = \sqrt{x^4+1}$, find $\dfrac{dy}{dx}$.

Answer: $\dfrac{2x^3}{\sqrt{x^4+1}}$

12. If $f(t) = \dfrac{t^2}{t^2+1}$, find $f'(t)$.

Answer: $\dfrac{2t}{(t^2+1)^2}$

13. If a function f is differentiable, then $\dfrac{d}{dx} f(\sqrt{x}) =$

Circle the appropriate answer: $f'(\sqrt{x})$ $\quad$ $f'(x)\dfrac{1}{2\sqrt{x}}$ $\quad$ $\boxed{f'(\sqrt{x})\dfrac{1}{2\sqrt{x}}}$

14. if $f(x) = \sin(|x|)$, find $f'(x)$ for $x \neq 0$.

Answer: $\cos(|x|)\dfrac{x}{|x|}$ OR $\cos(x)\dfrac{x}{|x|}$

15. $\dfrac{d}{dx}|x^2 - x| = |2x - 1|$

Circle the appropriate answer: True $\quad$ $\boxed{\text{False}}$

16. $\dfrac{d}{dx}(\sin^2 x) = -\dfrac{d}{dx}(\cos^2 x)$.

Circle the appropriate answer: $\boxed{\text{True}}$ $\quad$ False

17. Find the x-**coordinate** of **all points** on the curve $y = x^3 - 3x$ where the tangent line is horizontal.

    Answer: $x = \pm 1$

18. What is $\lim_{x \to \frac{\pi}{3}} \dfrac{\sin(x) - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{3}}$?

    Answer: $\dfrac{1}{2}$

19. What is $\lim_{x \to a} \dfrac{\cos(x^2) - \cos(a^2)}{x - a}$?

    Answer: $-2a\sin(a^2)$

20. What is $\lim_{h \to 0} \dfrac{\tan(3+h)^3 - \tan(27)}{h}$?

    Answer: $27\sec^2(27)$

21. There are no points on the curve $y = x^3 + 2x - 10$ where the tangent is horizontal.

    Circle the appropriate answer $\boxed{\text{True}}$ False

22. If a function f is continuous at $x = a$, then f is differentiable at $x = a$.

    Circle the appropriate answer Always True $\boxed{\text{Can be False}}$

Part B: For the following problems give a complete solution. Partial credit is possible and you must **SHOW ALL YOUR WORK.**

I) (a) If $f(x) = x\tan(x^2)$, find $f'(x)$.

By the Product Rule

$$f'(x) = x\frac{d}{dx}\tan(x^2) + \tan(x^2)\frac{d}{dx}x.$$

By the Chain Rule

$$f'(x) = x\sec^2(x^2)2x + \tan(x^2)(1).$$

$$f'(x) = 2x^2\sec^2(x^2) + \tan(x^2).$$

(b) If $g(x) = \sin^2(3x) + \sec(\sqrt{x});\;\; x > 0,\;$ find $g'(x)$.

By the Addition and the Chain Rule

$$g'(x) = \frac{d}{dx}\sin^2(3x) + \frac{d}{dx}\sec(\sqrt{x}) = 2\sin(3x)\frac{d}{dx}\sin(3x) + \sec(\sqrt{x})\tan(\sqrt{x})\frac{d}{dx}\sqrt{x}.$$

$$g'(x) = 2\sin(3x)\cos(3x)(3) + \sec(\sqrt{x})\tan(\sqrt{x})\frac{1}{2\sqrt{x}}.$$

$$g'(x) = 6\sin(3x)\cos(3x) + \sec(\sqrt{x})(\tan\sqrt{x})\frac{1}{2\sqrt{x}}.$$

(c) If $f(t) = \sqrt{1 + \sin^2(5t)};\;$ find $f'(t)$

By a repeated use of the the Chain Rule

$$f'(t) = \frac{1}{2}(1 + \sin^2(5t))^{-\frac{1}{2}}\frac{d}{dt}(1 + \sin^2(5t)).$$

$$f'(t) = \frac{1}{2}(1 + \sin^2(5t))^{-\frac{1}{2}}(2\sin(5t)\cos(5t)(5)).$$

$$f'(t) = \frac{5\sin(5t)\cos(5t)}{\sqrt{1 + \sin^2(5t)}}.$$

II) On a distant planet a ball is thrown vertically upward with a velocity of 100 ft/s. Its height after t seconds is $100t - 10t^2$.

(a) What is the **maximum height** reached by the ball?

The ball reaches its maximum height when $s'(t) = 0$.

$$s'(t) = 100 - 20t = 0 \Longleftrightarrow t = 5s.$$

The maximum height is

$$s(5) = 500 - 250 = 250ft.$$

(b) What is the velocity of the ball when it is **210 ft** above the ground on its way up? On its way down?

$$s(t) = 210 = 100t - 10t^2 \Longleftrightarrow 10t^2 - 100t + 210 = 0 \Longleftrightarrow$$

Simplifying by 10 we get

$$t^2 - 10t + 21 = (t-3)(t-7) = 0 \Longleftrightarrow t = 3s \text{ or } t = 7s.$$

The velocity of the ball on its way up is

$$s'(3) = 100 - 60 = 40ft/s.$$

The velocity of the ball on its way down is

$$s'(7) = 100 - 140 = -40ft/s.$$

(c) Compute $\lim_{\theta \to 0} \frac{1-\cos(4\theta)}{\theta^2}$.

$$\lim_{\theta \to 0} \frac{1-\cos(4\theta)}{\theta^2} = \lim_{\theta \to 0} \frac{(1-\cos(4\theta))}{\theta^2} \frac{(1+\cos(4\theta))}{(1+\cos(4\theta))} =$$

$$\lim_{\theta \to 0} \frac{\sin^2(4\theta)}{\theta^2} \frac{1}{(1+\cos(4\theta))} = \lim_{\theta \to 0} (\frac{\sin(4\theta)}{\theta})^2 \frac{1}{(1+\cos(4\theta))} =$$

$$\lim_{\theta \to 0} (4\frac{\sin(4\theta)}{4\theta})^2 \frac{1}{(1+\cos(4\theta))} = 16\frac{1}{2} = 8.$$

III) (a) Find the **linearization** of $\sqrt[3]{x}$ at $\mathbf{a} = \mathbf{27}$ and use it to approximate $\sqrt[3]{26}$.

Since $f(x) = (x)^{\frac{1}{3}}$, then $f(27) = 3$, and $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3}\frac{1}{x^{\frac{2}{3}}}$. It follows that $f'(27) = \frac{1}{27}$, and

$$L(x) = f(27) + f'(27)(x - 27) = 3 + \frac{1}{27}(x - 27).$$

$$\sqrt[3]{26} \approx L(26) = 3 + \frac{1}{27}(26 - 27) = 3 - \frac{1}{27}.$$

(b) If $y = \cos^2(5\theta) + \tan(3\theta)$, find $\frac{dy}{d\theta}$.

By the Addition and Chain Rules we have

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\cos^2(5\theta) + \frac{d}{d\theta}\tan(3\theta) =$$

$$2\cos(5\theta)\frac{d}{d\theta}\cos(5\theta) + \sec^2(3\theta)(3) = 2\cos(5\theta)(-\sin(5\theta)(5)) + 3\sec^2(3\theta) =$$

$$-10\sin(5\theta)\cos(5\theta) + 3\sec^2(3\theta).$$

(c) If $f(x) = \sin\left(\frac{\cos(x)}{x}\right)$, find $f'(x)$.

By the Chain Rule

$$f'(x) = \cos\left(\frac{\cos(x)}{x}\right)\frac{d}{dx}\left(\frac{\cos(x)}{x}\right).$$

By the quotient Rule we then have

$$f'(x) = \cos\left(\frac{\cos(x)}{x}\right)\frac{x(-\sin(x)) - \cos(x)(1)}{x^2} = -\frac{x(\sin(x)) + \cos(x)(1)}{x^2}\cos\left(\frac{\cos(x)}{x}\right).$$

IV) (a) The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the **surface area** increasing when the length of an edge is 30 cm?

Let $x(t)$ be the edge at time t.

The volume $V(t) = (x(t))^3$ and the surface area $S(t) = 6\,(x(t))^2$.

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \quad \text{and} \quad \frac{dS}{dt} = 12x\frac{dx}{dt}.$$

$$10 = 3x^2\frac{dx}{dt} \Longrightarrow \frac{dx}{dt} = \frac{10}{3x^2}.$$

It follows that

$$\frac{dS}{dt} = 12x\frac{10}{3x^2} = \frac{40}{x}.$$

So when $x = 10$

$$\frac{dS}{dt} = \frac{4}{3}cm^2/min.$$

(b) Compute $\lim_{\theta\to 0} \dfrac{\sin(3\theta) + \sin(\theta)}{\theta + \sin(\theta)\cos(\theta)}$

$$\lim_{\theta\to 0} \frac{\sin(3\theta) + \sin(\theta)}{\theta + \sin(\theta)\cos(\theta)} = \lim_{\theta\to 0} \frac{\frac{\sin(3\theta)}{\theta} + \frac{\sin(\theta)}{\theta}}{\frac{\theta}{\theta} + \frac{\sin(\theta)}{\theta}\cos(\theta)} =$$

$$\lim_{\theta\to 0} \frac{3\frac{\sin(3\theta)}{3\theta} + \frac{\sin(\theta)}{\theta}}{\frac{\theta}{\theta} + \frac{\sin(\theta)}{\theta}\cos(\theta)} = \frac{3+1}{1+1\times 1} = 2.$$

(c) Compute $\lim_{x\to 2} \dfrac{\sqrt{x^2+1} - \sqrt{5}}{x-2}$

$$\lim_{x\to 2} \frac{\sqrt{x^2+1} - \sqrt{5}}{x-2} = f'(2), \quad \text{where} \ \ f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}.$$

Since by the Chain Rule

$$f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+1}},$$

we get

$$\lim_{x\to 2} \frac{\sqrt{x^2+1} - \sqrt{5}}{x-2} = f'(2) = \frac{2}{\sqrt{5}}.$$

V) (a) (10 points) Find the linear approximation of the function $f(x) = \sqrt{9+x}$ at $a = 0$ and use it to approximate the number $\sqrt{9.01}$.

We have $f(x) = \sqrt{9+x} = (9+x)^{\frac{1}{2}}$.

Since

$$f'(x) = \frac{1}{2}(9+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{9+x}},$$

we get

$$f(0) = 3, \quad f'(0) = \frac{1}{6}, \quad \text{and}$$

$$L(x) = 3 + \frac{1}{6}x.$$

$$\sqrt{9.01} = \sqrt{9+.01} \approx L(0.01) = 3 + \frac{1}{6}\frac{1}{100} = 3 + \frac{1}{600}.$$

(b) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is **1m higher** than the bow of the boat. If the rope is pulled in at a **rate of 1m/s**, **how fast is the boat approaching the dock** when it is **8m from the dock**?

Let $X(t)$ be the distance from the boat to the dock at time t, and let $Y(t)$ be the amount of rope out at time t. We are given that

$\dfrac{dY}{dt} = -1$, and we need $\dfrac{dX}{dt}$ when $X = 8m$. Since

$$Y^2 = X^2 + 1,$$

differentiating with respect to time, we get

$$2YY' = 0 + 2XX' \text{ which implies } X' = \frac{YY'}{X}.$$

When $X = 8, Y^2 = 1 + 64$, then $Y = \sqrt{65}$. It follows that when $X = 8m$,

$$\frac{dX}{dt} = \frac{\sqrt{65}(-1)}{8} = -\frac{\sqrt{65}}{8}.$$

The boat is approching the dock at the rate of $\dfrac{\sqrt{65}}{8} m/s$.

VI) (a) Find the equation of the **tangent line** to the curve $xy+x^3 = y^2$ at the point $(2,-2)$.

Differentiating with respect to x, we get

$$\frac{d}{dx}(xy+x^3) = \frac{d}{dx}y^2$$

$$xy' + y(1) + 3x^2 = 2yy'.$$

Solving for y', we get

$$2yy' - xy' = 3x^2 + y$$

$$y'(2y-x) = 3x^2 + y \text{ and } y' = \frac{3x^2+y}{2y-x};\ 2y - x \neq 0.$$

At the point $(2,-2)$, $y' = \dfrac{12-2}{-4-2} = -\dfrac{5}{3}$. The equation of the tangent line is

$$y + 2 = -\frac{5}{3}(x-2)$$

(b) Find the points on the curve $x^2 + xy + y^2 = 3$, where the tangent line is horizontal.

Differentiating with respect to x, we get

$$\frac{d}{dx}(x^2+xy+y^2) = \frac{d}{dx}3$$

$$2x + xy' + y + 2yy' = 0.$$

Solving for y', we get

$$2yy' + xy' = -2x - y$$

$$y'(2y+x) = -2x - y \text{ and } y' = \frac{-2x-y}{2y+x};\ 2y + x \neq 0.$$

It follows that $y' = 0 \iff -2x - y = 0 \iff y = -2x$.
Substituting $y = -2x$ in $x^2 + xy + y^2 = 3$, we get

$$x^2 - 2x^2 + 4x^2 = 3 \iff 3x^3 = 3 \iff x^2 = 1.$$

Thus $x = \pm 1$, and the points on the curve $x^2 + xy + y^2 = 3$ where the tangent line is horizontal are

$$(1,-2) \text{ and } (-1,2)$$

(c) Find the equation to the tangent line to the curve $x^3 - 2xy + y^4 = \sin(y-1)$ at the point $(1,1)$.

It is obvious that the point $(1,1)$ is on the curve $x^3 - 2xy + y^4 = \sin(y-1)$ Differentiating with respect to x, we get

$$\frac{d}{dx}(x^3 - 2xy + y^4) = \frac{d}{dx}\sin(y-1)$$
$$3x^2 - 2xy' - 2y + 4y^3y' = \cos(y-1)y'.$$

At the point $(1,1)$, we have

$$3 - 2 - 2 + 4y' = y' \Longrightarrow 3y' = 1 \Longleftrightarrow y' = \frac{1}{3}.$$

The equation of the tangent line at $(1,1)$ is

$$y - 1 = \frac{1}{3}(x-1).$$

VII) (a) Find the equation of the tangent line from the point $(0,1)$ to the curve $y = \dfrac{1}{x}$.

Note that $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$, $x \neq 0$. Let $A = (0,1)$, and $T(a, \dfrac{1}{a})$ be a tangency point on the curve. Then the slope of $AT = \dfrac{\frac{1}{a} - 1}{a - 0} = -\dfrac{1}{a^2}$. This implies

$$\frac{1-a}{a^2} = -\frac{1}{a^2} \Longleftrightarrow 1 - a = -1 \Longleftrightarrow a = 2.$$

So the tangency point is $T(2, \dfrac{1}{2})$, the slope of the tangent line is $m_{tan} = -\dfrac{1}{4}$, and the equation of th tangent line is

$$y - \frac{1}{2} = -\frac{1}{4}(x-2).$$

(b) Show that the family of circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ are orthogonal.

It is clear that the y-axis is tangent to $x^2 + y^2 = ax$ at $(0,0)$, and that the x-axis is tangent to $x^2 + y^2 = by$ at $(0,0)$. Thus the families of circles is orthogonal at $(0,0)$, which is one of its intersection points. Let $(x_0, y_0) \neq (0,0)$ be another intersection point then we have

$$x_0^2 + y_0^2 = ax_0 \quad \text{and} \quad x_0^2 + y_0^2 = by_0.$$

Let us find the respective slopes of the tangent lines at (x_0, y_0).
Differentiating $x^2 + y^2 = ax$ with respect to x, we get

$$2x + 2yy' = a, \Longrightarrow y' = \frac{a - 2x}{2y}.$$

Differentiating $x^2 + y^2 = by$ with respect to x, we get

$$2x + 2yy' = by', \implies y' = \frac{2x}{b - 2y}.$$

At an intersection point (x_0, y_0), the slopes of the tangent lines are:

$$m_a = \frac{a - 2x_0}{2y_0} \quad \text{and} \quad m_b = \frac{2x_0}{b - 2y_0}$$

Since

$$m_a.m_b = \frac{a - 2x_0}{2y_0} \times \frac{2x_0}{b - 2y_0} = \frac{2ax_0 - 4x_0^2}{2by_0 - 4y_0^2} = \frac{2x_0^2 + 2y_0^2 - 4x_0^2}{2x_0^2 + 2y_0^2 - 4y_0^2} = \frac{2y_0^2 - 2x_0^2}{2x_0^2 - 2y_0^2} = -1,$$

the families of circles are orthogonal at the intersection point (x_0, y_0).

(c) Show that the parabola $x = y^2$ and the ellipse $2x^2 + y^2 = 3$ are orthogonal.

Let (x_0, y_0) be an intersection point, then we have

$$x_0 = y_0^2 \text{ and } 2x_0^2 + y_0^2 = 3.$$

Differentiating $x = y^2$ with respect to x, we get

$$1 = 2yy', \implies y' = \frac{1}{2y}.$$

Differentiating $2x^2 + y^2 = 3$ with respect to x, we get

$$4x + 2yy' = 0, \implies y' = -\frac{2x}{y}.$$

At an intersection point (x_0, y_0), the slopes of the tangent lines are:

$$m_{parabola} = \frac{1}{2y_0} \quad \text{and} \quad m_{ellipse} = -\frac{2x_0}{y_0}$$

Since

$$m_{parabola}.m_{ellipse} = \frac{1}{2y_0} \times -\frac{2x_0}{y_0} = -\frac{2x_0}{2y_0^2} = -1,$$

the parabola and the ellipse are orthogonal at any intersection point (x_0, y_0).

VIII) (a) The cost function for production of a commoditity is

$$C(x) = 2000 + 100x - 0.1x^2.$$

1) Find the **average cost per machine** of producing the **first 100 washing machines**.

$$\frac{C(100) - C(0)}{100 - 0} = \frac{2000 + 10000 - 1000 - 2000}{100} = 90 \text{ dollars.}$$

2) Find the **marginal cost** when 100 machines are produced.
$C'(x) = 100 - 0.2x^2$. Hence $C'(100) = 100 - 20 = 80$.

3) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more machine after the first 100 have been made, by calculating the later cost directly.

$$C(101) - C(100) = 2000 + 100 \times 101 - 0.1 \times 101^2 - (2000 - 100 \times 100 - 0.1 \times 100^2) = 79.9$$

Hence

$$C(101) - C(100) \approx C'(100).$$

IX) (a) A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2m^3$/min, find the rate at which the water level is rising when the water is 3m deep.

Let $V(t), h(t)$, and $R(t)$ be the volume, the height and the radius respectively, at time t.
We have $\dfrac{dV}{dt} = 2m^3/\text{min}$, we need $\dfrac{dh}{dt}$ when $h = 3\text{m}$.
$V(t) = \dfrac{\pi}{3}R^2(t)h(t)$ and

$$\frac{r}{h} = \frac{2}{4} \Longleftrightarrow r = \frac{1}{2}h.$$

Then $V = \dfrac{\pi}{3}\dfrac{h^3}{4} = \dfrac{\pi}{12}h^3$. Differentiating with respect to t, we get

$$\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \Longrightarrow$$

$$2 = \frac{\pi}{4}3^2\frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{8}{9\pi}.$$

The water is rising at the rate of $\dfrac{8}{9\pi}$ m/min.

(b) Bike A is traveling **west** at 10 mi/h and bike B is traveling **north** at 12 mi/h. Bike A is headed **toward** the intersection of the two roads while bike B is headed **away** from the intersection of the two roads. At what rate is the distance between bike A and bike B changing when bike A is 6 mi and bike B is 8 mi from the intersection?

Let $X(t)$ be the distance from bike A to the intersection, $Y(t)$ be the distance from bike B to the intersection, and let $Z(t)$ be the distance between Bike A and Bike B at time t. We have

$$\frac{dX}{dt} = -10, \quad \text{and} \quad \frac{dY}{dt} = 12.$$

We need $\dfrac{dZ}{dt}$ when $X = 6$ mi, and $Y = 8$ mi from the intersection. Since

$$Z = \sqrt{X^2 + Y^2}$$

Differentiating with respect to we get

$$Z' = \frac{1}{2}(X^2 + Y^2)^{-\frac{1}{2}}(2XX' + 2YY') = \frac{XX' + YY'}{\sqrt{X^2 + Y^2}}.$$

When $X = 6$ mi, and $Y = 8$ mi

$$Z' = \frac{6(-10) + 8(12)}{\sqrt{36 + 64}} = 3.6.$$

The distance between the two bikes is increasing at the rate of 3.6mi/h