Sample Questions Exam I, FS2009

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Calculators are neither needed nor allowed.

Part A: (SHORT ANSWER QUESTIONS) Do the following problems. Write the answer in the space provided. Only the answers will be graded; **there is no partial credit.**

1. What is the domain of the function $f(x) = \dfrac{1}{\sqrt{8-5x}}$?

   Answer: $x < \dfrac{8}{5}$ or $\left(-\infty, \dfrac{8}{5}\right)$

2. Determine whether the function $f(x) = x^2 \sin(x) + x^3$ is **even**, **odd** or **neither**.
   Answer: odd

3. Compute $\lim\limits_{x \to 1^+} \dfrac{\sqrt{x-1}+x}{1-2x}$
   Answer: -1

4. The composition of an even function and an odd function is an even function.

   Circle the appropriate answer: $\boxed{\text{Always true}}$ Can be false

5. If $f(x) = x^4 + 1$ and $g(x) = \dfrac{1}{\sqrt{x-1}}$, what is $g(f(x))$?

   Answer: $\dfrac{1}{x^2}$ or $\dfrac{1}{\sqrt{x^4}}$

6. Express the **surface area** A of a **cube** as a function of **its volume** V.
Answer: $A = 6(\sqrt[3]{V})^2$

7. Compute $\lim_{x \to 4} \frac{x^2 - 3x - 4}{x^2 - 6x + 8}$.
Answer: $\frac{5}{2}$

8. Compute $\lim_{x \to 3^-} \frac{2(x-3)}{|x-3|}$
Answer: -2

9. $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ is:

Circle the appropriate answer: 0 ∞ does not exist

10. Compute $\lim_{x \to 5^-} \frac{x-6}{5-x}$
Answer: $-\infty$

11. If $f(2) = 5$, $g(2) = 3$, $f'(2) = 6$ and $g'(2) = 1$, find $\left(\frac{f}{g}\right)'(2)$.

Answer: $\dfrac{3.6 - 5.1}{9} = \dfrac{18 - 5}{9} = \boxed{\dfrac{13}{9}}$

12. If $f(2) = 5$, $g(2) = 3$, $f'(2) = 6$ and $g'(2) = 1$, find $(f.g)'(2)$.
Answer: $5.1 + 3.6 = 5 + 18 = 23.$

13. Find the equation of the **tangent line** to the curve $y = x^3$ at the point $(2, 8)$.
Answer: $y - 8 = 12(x - 2).$

14. If f is a continuous function on $[0, 1]$, $f(0) < 0$ and $f(1) > 0$, then there is $0 < c < 1$ such that $f(c) = 0$.

Circle the appropriate answer $\boxed{\text{Always True}}$ Can be False

15. Find $\lim_{\theta \to 0} \dfrac{\sin(5\theta)}{\sin(8\theta)}$.
Answer: $\dfrac{5}{8}$

16. If the tangent line to $y = f(x)$ at $(3, 5)$ passes through the point $(1, 4)$ find $f'(3)$.
Answer: $\dfrac{5-4}{3-1} = \boxed{\dfrac{1}{2}}$

17. Find $\lim_{x \to 1} \dfrac{x^9 - 1}{x - 1}$.
Answer: 9

18. **True or False.** If $h(x) = x^5$, then

$$\lim_{x \to 2} \frac{h(x) - h(2)}{x - 2} = 80.$$

Answer: True

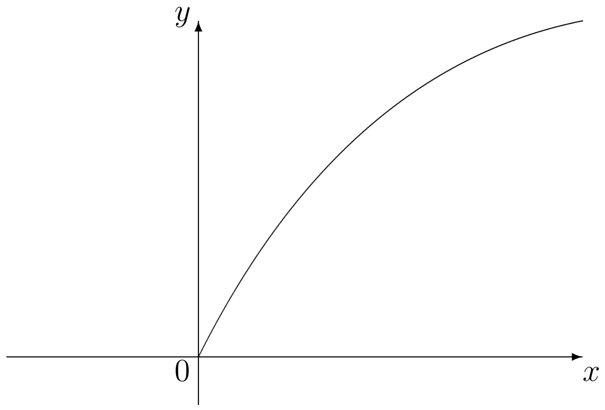
19. Find the x-**coordinate** of **all points** on the curve $y = x^3 - 3x$ where the tangent line is horizontal.
Answer: $x = 1$ or $x = -1$.

20. If a function f is continuous at $x = a$, then f is differentiable at $x = a$.

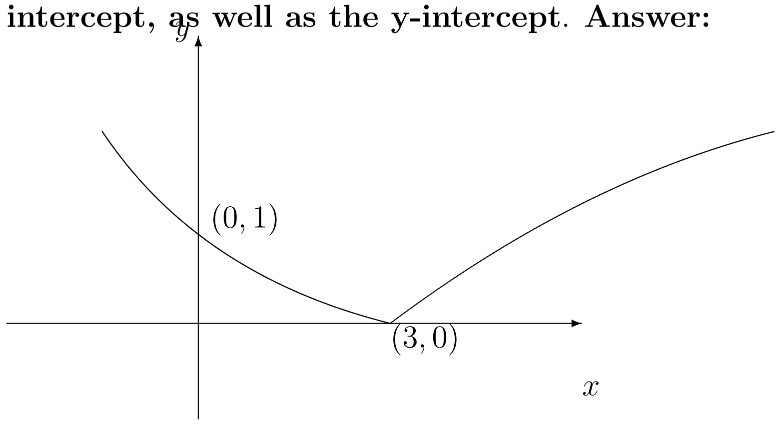
Circle the appropriate answer Always True $\boxed{\text{Can be False}}$

Part B: For the following problems give a complete solution. Partial credit is possible and you must **SHOW ALL YOUR WORK.**

I) (a) Sketch the graph of the function $y = \sqrt{x}$. **Make sure to label the x-axis, as well as the y-axis. Answer:**



(b) Sketch the graph of the function $y = |\sqrt{x+1} - 2|$. **Make sure to label the x-intercept, as well as the y-intercept. Answer:**



II) (a) If $f(x) = \sqrt{4+3x}$ and $g(x) = \dfrac{x^2-4}{3}$, find $f \circ g$ and $g \circ f$, and **make sure to specify their domain.** **Answer:** $f \circ g(x) = f\left(\dfrac{x^2-4}{3}\right) = \sqrt{4+x^2-4} = \sqrt{x^2}$

Hence, $f \circ g(x) = |x|;\ -\infty < x < \infty$

$g \circ f(x) = g(\sqrt{4+3x}) = \dfrac{4+3x-4}{4} = x$

Hence, $g \circ f(x) = x;\ -\dfrac{4}{3} \leq x$

(b) **Use the Squeeze Theorem** to explain why $\lim_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right) = 0$.

Answer: Since $-1 \leq \cos\left(\dfrac{1}{x}\right) \leq 1$ for all $x \neq 0$,

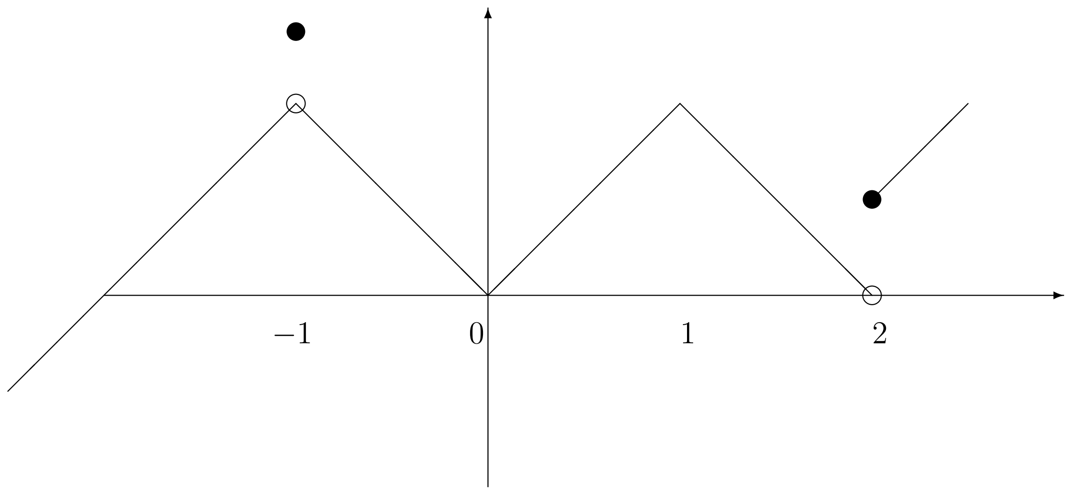
then $-x^2 \leq x^2 \cos\left(\dfrac{1}{x}\right) \leq x^2$ for all $x \neq 0$.

Since $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$

By the Squeeze Theorem

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

III) The graph of a function f is given below.



(a) At what numbers is f discontinuous? Give reasons for your answers.
Answer: The function f is discontinuous at $x = -1$ and at $x = 2$.
At $x = -1$, $\lim_{x \to -1} f(x) = 1 \neq f(-1)$. It is a removable discontinuiuty.
At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist because

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x).$$

At $x = 2$ we have a jump discontinuity.

(b) At what numbers is f not differentiable? Give reasons for your answers.
Answer: The function f is not differentiable at $x = -1$,and $x = 2$, because f is not even continuous there.
It is not differentiable at $x = 0$, and $x = 1$, because the graph is not smooth there.

IV) (a) If

$$g(x) = \begin{cases} \dfrac{x^2 - x}{x - 1} & \text{if } x > 1 \\ \sin(x-1) + 2 & \text{if } x \leq 1 \end{cases}$$

(1) Find $\lim_{x \to 1^+} g(x)$

(2) Find $\lim_{x \to 1^-} g(x)$

(3) Is the function g continuous at $x = 1$? Explain.

Answer:

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \frac{x^2 - x}{x - 1} = \lim_{x \to 1^+} \frac{x(x-1)}{x-1} = 1$$

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \sin(x-1) + 2 = \sin 0 + 2 = 2$$

Since $\lim_{x \to 1^+} g(x) \neq \lim_{x \to 1^-} g(x)$, then
$\lim_{x \to 1} g(x)$ does not exist and g is discontinuous at $x = 1$.

(b) Compute $\lim_{t \to 0} \dfrac{t}{\sqrt{1 + 5t} - 1}$

Answer:

$$\lim_{t \to 0} \frac{t}{\sqrt{1+5t} - 1} = \lim_{t \to 0} \frac{t}{\sqrt{1+5t} - 1} \cdot \frac{\sqrt{1+5t} + 1}{\sqrt{1+5t} + 1}$$

$$= \lim_{t \to 0} \frac{t(\sqrt{1+5t} + 1)}{1 + 5t - 1} = \lim_{t \to 0} \frac{\sqrt{1+5t} + 1}{5}$$

$$= \frac{2}{5}$$

V) (a) (8 points) Find the equation of the **tangent line** to the curve $y = \sqrt{x}$ that is parallel to the line $x - 4y = 1$.

Answer: The slope of the line $x - 4y = 1$ is $\frac{1}{4}$.
Since the slope of the tangent line to $y = \sqrt{x}$ is

$$y' = \frac{1}{2\sqrt{x}}$$

setting

$$\frac{1}{2\sqrt{x}} = \frac{1}{4}$$

We get

$$\sqrt{x} = 2.$$

Which implies $x = 4$. Thus the tangency point is $(4, 2)$,
and the equation of the tangent line is
$y - 2 = \frac{1}{4}(x - 4)$.

VI) On a distant planet a ball is thrown vertically upward with a velocity of 100 ft/s. Its height after t seconds is $100t - 10t^2$.

(a) What is the **maximum height** reached by the ball?
Answer:

$$s(t) = 100t - 10t^2$$
$$v(t) = \frac{ds}{dt} = 100 - 20t$$

the ball reaches its maximum height when $v(t) = 0$

$$v(t) = 100 - 20t = 0 \Leftrightarrow t = 5\,\text{sec}$$

the maximum height $= 100(5) - 10.25 = 250\text{ft}$

(b) What is the velocity of the ball when it is **210 ft** above the ground on its way up? On its way down?
Answer:

$$s(t) = 100t - 10t^2 = 210 \Leftrightarrow$$
$$10t^2 - 100t + 210 = 0 \Leftrightarrow$$
$$t^2 - 10t + 21 = 0 \Leftrightarrow (t-3)(t-7) = 0$$
$$t = 3\,\text{sec} \quad \text{or} \quad t = 7\,\text{sec}$$
$$v(3) = \text{ velocity on the way up } = 100 - 20(3) = 40\text{ft/sec}$$
$$v(7) = \text{ velocity on the way down } = 100 - 20(7) = -40\text{ft/sec}$$

VII Compute $\lim_{\theta \to 0} \frac{1-\cos(4\theta)}{\theta^2}$.

Answer:

$$\lim_{\theta \to 0} \frac{1-\cos(4\theta)}{\theta^2} = \lim_{\theta \to 0} \frac{1-\cos(4\theta)}{\theta^2} \cdot \frac{1+\cos(4\theta)}{1+\cos(4\theta)}$$

$$= \lim_{\theta \to 0} \frac{1-\cos^2(4\theta)}{\theta^2} \cdot \frac{1}{1+\cos(4\theta)}$$

$$= \lim_{\theta \to 0} \left(\frac{\sin(4\theta)}{\theta} \right)^2 \cdot \frac{1}{1+\cos(4\theta)} = \lim_{\theta \to 0} \left(4 \frac{\sin(4\theta)}{4\theta} \right)^2 \cdot \lim_{\theta \to 0} \frac{1}{1+\cos(4\theta)}$$

$$= 16 \cdot \frac{1}{2} = 8$$

VIII) (a) Find the **points** on the curve $y = \dfrac{x}{x-1}$ where the **tangent lines are parallel** to the line $x + y = 4$.

Answer:

$$\frac{dy}{dx} = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

the slope of the line $x + y = 4$ is -1.

$$\frac{-1}{(x-1)^2} = -1 \Leftrightarrow (x-1)^2 = 1 \Leftrightarrow$$

$x - 1 = 1$ or $x - 1 = -1 \Leftrightarrow$

$x = 2$ or $x = 0$

the points are $(2, 2)$ and $(0, 0)$.

(b) Use the definition given in Section 3.1 to find the slope of the tangent line to the graph of $f(x) = \dfrac{1}{x}$ at the point $\left(2, \dfrac{1}{2}\right)$. Then find an equation of the tangent line there.

Answer:

$$\begin{aligned} m_{\tan} &= \text{ slope of the tangent line at } \left(2, \frac{1}{2}\right) \\ &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\ &= \lim_{h \to 0} \frac{2 - (2+h)}{2(2+h)h} \\ &= \lim_{h \to 0} \frac{-h}{2(2+h)h} \\ &= \lim_{h \to 0} \frac{-1}{2(2+h)} = -\frac{1}{4} \end{aligned}$$

the equation of the tangent line at $\left(2, \dfrac{1}{2}\right)$ is

$y - \dfrac{1}{2} = -\dfrac{1}{4}(x - 2)$ or

$y = -\dfrac{1}{4}x + 1.$

IX) (a) (8 points) Show that there are **two tangent lines** to the curve $y = x^2$ through $(2,3)$. Find an equation of each of these lines.

Answer:

Let $T(a, a^2)$ be a tangency point, and let A denote the point$(2,3)$.

then slope of $AT = \dfrac{a^2 - 3}{a - 2}$

and since $\dfrac{dy}{dx} = 2x$

slope of $AT = 2a$

Hence $\dfrac{a^2 - 3}{a - 2} = 2a \Leftrightarrow$

$a^2 - 3 = 2a^2 - 4a \Leftrightarrow a^2 - 4a + 3 = (a-1)(a-3) = 0 \Leftrightarrow$

$a = 1$ or $a = 3$.

The tangency points are $(1,1)$ and $(3,9)$ the slopes of the tangent lines are $m_1 = 2$ and $m_2 = 6$ the equation of the tangent lines are

$y - 1 = 2(x - 1)$ and

$y - 9 = 6(x - 3)$