SOCIETY OF ACTUARIES

EXAM M ACTUARIAL MODELS

EXAM M SAMPLE SOLUTIONS

Some of the questions in this study note are taken from past SOA examinations.

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Question #1

Key: E

$$ {}_{2|}q_{\overline{30:34}} = {}_2p_{\overline{30:34}} - {}_3p_{\overline{30:34}} $$

$$ {}_2p_{30} = (0.9)(0.8) = 0.72 $$

$$ {}_2p_{34} = (0.5)(0.4) = 0.20 $$

$$ {}_2p_{30:34} = (0.72)(0.20) = 0.144 $$

$$ {}_2p_{\overline{30:34}} = 0.72 + 0.20 - 0.144 = 0.776 $$

$$ {}_3p_{30} = (0.72)(0.7) = 0.504 $$

$$ {}_3p_{34} = (0.20)(0.3) = 0.06 $$

$$ {}_3p_{30:34} = (0.504)(0.06) = 0.03024 $$

$$ {}_3p_{\overline{30:34}} = 0.504 + 0.06 - 0.03024 $$

$$ = 0.53376 $$

$$ {}_{2|}q_{\overline{30:34}} = 0.776 - 0.53376 $$

$$ = 0.24224 $$

Alternatively,

$$ {}_{2|}q_{\overline{30:34}} = {}_{2|}q_{30} + {}_{2|}q_{34} - {}_{2|}q_{30:34} $$

$$ = {}_2p_{30}q_{32} + {}_2p_{34}q_{36} - {}_2p_{30:34}(1 - p_{32:36}) $$

= (0.9)(0.8)(0.3) + (0.5)(0.4)(0.7) – (0.9)(0.8)(0.5)(0.4) [1-(0.7)(0.3)]
= 0.216 + 0.140 – 0.144(0.79)
= 0.24224

Alternatively,

$$ {}_{2|}q_{\overline{30:34}} = {}_3q_{30} \times {}_3q_{34} - {}_2q_{30} \times {}_2q_{34} $$

$$ = (1 - {}_3p_{30})(1 - {}_3p_{34}) - (1 - {}_2p_{30})(1 - {}_2p_{34}) $$

$$ = (1 - 0.504)(1 - 0.06) - (1 - 0.72)(1 - 0.20) $$

$$ = 0.24224 $$

(see first solution for ${}_2p_{30}$, ${}_2p_{34}$, ${}_3p_{30}$, ${}_3p_{34}$)

Question #2

Key: E

$$1000\bar{A}_x = 1000\left[\bar{A}^{1}_{x:\overline{10}|} + {}_{10|}\bar{A}_x\right]$$

$$= 1000\left[\int_0^{10} e^{-0.04t}e^{-0.06t}(0.06)dt + e^{-0.4}e^{-0.6}\int_0^{\infty} e^{-0.05t}e^{-0.07t}(0.07)dt\right]$$

$$= 1000\left[0.06\int_0^{10} e^{-0.1t}dt + e^{-1}(0.07)\int_0^{\infty} e^{-0.12t}dt\right]$$

$$= 1000\left[0.06\left[\frac{-e^{-0.10t}}{0.10}\right]_0^{10} + e^{-1}(0.07)\left[\frac{-e^{-0.12t}}{0.12}\right]_0^{\infty}\right]$$

$$= 1000\left[\frac{0.06}{0.10}\left[1-e^{-1}\right] + \frac{0.07}{0.12}e^{-1}\left[1-e^{-1.2}\right]\right]$$

$$= 1000(0.37927 + 0.21460) = 593.87$$

Because this is a timed exam, many candidates will know common results for constant force
and constant interest without integration.

For example $\bar{A}^{1}_{x:\overline{10}|} = \dfrac{\mu}{\mu+\delta}\left(1 - {}_{10}E_x\right)$

$${}_{10}E_x = e^{-10(\mu+\delta)}$$

$$\bar{A}_x = \frac{\mu}{\mu+\delta}$$

With those relationships, the solution becomes

$$1000\bar{A}_x = 1000\left[\bar{A}^{1}_{x:\overline{10}|} + {}_{10}E_x\ A_{x+10}\right]$$

$$= 1000\left[\left(\frac{0.06}{0.06+0.04}\right)\left(1-e^{-(0.06+0.04)10}\right) + e^{-(0.06+0.04)10}\left(\frac{0.07}{0.07+0.05}\right)\right]$$

$$= 1000\left[(0.60)\left(1-e^{-1}\right) + 0.5833e^{-1}\right]$$

$$= 593.86$$

Question #3
Key: A

$$B = \begin{cases} c(400-x) & x < 400 \\ 0 & x \geq 400 \end{cases}$$

$$100 = E(B) = c\bullet 400 - cE(X \wedge 400)$$

$$= c\bullet 400 - c\bullet 300\left(1 - \frac{300}{300+400}\right)$$

$$= c\left(400 - 300\bullet\frac{4}{7}\right)$$

$$c = \frac{100}{228.6} = 0.44$$

Question #4

Key: C

Let $N = \#$ of computers in department
Let *X* = cost of a maintenance call
Let *S* = aggregate cost

$$\text{Var}(X) = [\text{Standard Deviation } (X)]^2 = 200^2 = 40{,}000$$

$$E(X^2) = \text{Var}(X) + [E(X)]^2$$

$$= 40{,}000 + 80^2 = 46{,}400$$

$$E(S) = N \times \lambda \times E(X) = N \times 3 \times 80 = 240N$$

$$\text{Var}(S) = N \times \lambda \times E(X^2) = N \times 3 \times 46{,}400 = 139{,}200N$$

$$\text{We want } \quad 0.1 \geq \Pr(S > 1.2E(S))$$

$$\geq \Pr\left(\frac{S - E(S)}{\sqrt{139{,}200N}} > \frac{0.2E(S)}{\sqrt{139{,}200N}}\right) \Rightarrow \frac{0.2 \times 240N}{373.1\sqrt{N}} \geq 1.282 = \Phi(0.9)$$

$$N \geq \left(\frac{1.282 \times 373.1}{48}\right)^2 = 99.3$$

Question #5

Key: B

$$\mu_x^{(\tau)} = \mu_x^{(1)} + \mu_x^{(2)} + \mu_x^{(3)} = 0.0001045$$
$$_t p_x^{(\tau)} = e^{-0.0001045t}$$

$$\text{APV Benefits} = \int_0^\infty e^{-\delta t} 1{,}000{,}000 \, _t p_x^{(\tau)} \mu_x^{(1)} dt$$
$$+\int_0^\infty e^{-\delta t} 500{,}000 \, _t p_x^{(\tau)} \mu_x^{(2)} dt$$
$$+\int_0^\infty e^{-\delta \tau} 200{,}000 \, _t p_x^{(\tau)} \mu_x^{(3)} dt$$
$$= \frac{1{,}000{,}000}{2{,}000{,}000} \int_0^\infty e^{-0.0601045t} dt + \frac{500{,}000}{250{,}000} \int_0^\infty e^{-0.0601045t} dt + \frac{250{,}000}{10{,}000} \int_0^\infty e^{-0.0601045t} dt$$
$$= 27.5(16.6377) = 457.54$$

Question #6

Key: B

$$APV \text{ Benefits} = 1000 A^1_{40:\overline{20|}} + \sum_{k=20}^{\infty} {}_k E_{40} 1000 v q_{40+k}$$

$$APV \text{ Premiums} = \pi \ddot{a}_{40:\overline{20|}} + \sum_{k=20}^{\infty} {}_k E_{40} 1000 v q_{40+k}$$

Benefit premiums $\Rightarrow$ Equivalence principle $\Rightarrow$

$$1000 A^1_{40:\overline{20|}} + \sum_{k=20}^{\infty} {}_k E_{40} 1000 v q_{40+k} = \pi \ddot{a}_{40:\overline{20|}} + \sum_{20}^{\infty} {}_k E_{40} 1000 v q_{40+k}$$

$$\pi = 1000 A^1_{40:\overline{20|}} / \ddot{a}_{40:\overline{20|}}$$
$$= \frac{161.32 - (0.27414)(369.13)}{14.8166 - (0.27414)(11.1454)}$$
$$= 5.11$$

While this solution above recognized that $\pi = 1000 P^1_{40:\overline{20|}}$ and was structured to take advantage of that, it wasn't necessary, nor would it save much time. Instead, you could do:

$$APV \text{ Benefits } = 1000 A_{40} = 161.32$$

$$APV \text{ Premiums} = \pi \ddot{a}_{40:\overline{20}|} + {}_{20}E_{40} \sum_{k=0}^{\infty} {}_kE_{60} 1000 v q_{60+k}$$
$$= \pi \ddot{a}_{40:\overline{20}|} + {}_{20}E_{40} 1000 A_{60}$$
$$= \pi \left[14.8166 - (0.27414)(11.1454) \right] + (0.27414)(369.13)$$
$$= 11.7612\pi + 101.19$$
$$11.7612\pi + 101.19 = 161.32$$
$$\pi = \frac{161.32 - 101.19}{11.7612} = 5.11$$

Question #7

Key: C

$$A_{70} = \frac{\delta}{i} \bar{A}_{70} = \frac{\ln(1.06)}{0.06}(0.53) = 0.5147$$
$$\ddot{a}_{70} = \frac{1 - A_{70}}{d} = \frac{1 - 0.5147}{0.06/1.06} = 8.5736$$
$$\ddot{a}_{69} = 1 + v p_{69} \ddot{a}_{70} = 1 + \left(\frac{0.97}{1.06} \right)(8.5736) = 8.8457$$
$$\ddot{a}_{69}^{(2)} = \alpha(2)\ddot{a}_{69} - \beta(2) = (1.00021)(8.8457) - 0.25739$$
$$= 8.5902$$

Note that the approximation $\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{(m-1)}{2m}$ works well (is closest to the exact answer, only off by less than 0.01). Since *m* = 2, this estimate becomes $8.8457 - \frac{1}{4} = 8.5957$

Question #8
Key: C

The following steps would do <u>in this multiple-choice context</u>:

1. From the answer choices, this is a recursion for an insurance or pure endowment.
2. Only C and E would satisfy $u(70) = 1.0$.
3. It is <u>not</u> E. The recursion for a pure endowment is simpler: $u(k) = \frac{1+i}{p_{k-1}} u(k-1)$
4. Thus, it must be C.

More rigorously, transform the recursion to its backward equivalent,
$u(k-1)$ in terms of $u(k)$:

$$u(k) = -\left(\frac{q_{k-1}}{p_{k-1}}\right) + \left(\frac{1+i}{p_{k-1}}\right) u(k-1)$$
$$p_{k-1} u(k) = -q_{k-1} + (1+i)\, u(k-1)$$
$$u(k-1) = vq_{k-1} + vp_{k-1}\, u(k)$$

This is the form of (a), (b) and (c) on page 119 of Bowers with $x = k-1$. Thus, the recursion could be:

$$A_x = vq_x + vp_x A_{x+1}$$

or $$A^1_{x:\overline{y-x|}} = vq_x + vp_x \;\; A^{\;1}_{x+1:\overline{y-x-1|}}$$

or $$A_{x:\overline{y-x|}} = vq_x + vp_x \;\; A_{x+1:\overline{y-x-1|}}$$

Condition (iii) forces it to be answer choice C

$u(k-1) = A_x$ fails at $x = 69$ since it is not true that

$$A_{69} = vq_{69} + (vp_{69})(1)$$

$u(k-1) = A^1_{x:\overline{y-x|}}$ fails at $x = 69$ since it is not true that

$$A^1_{69:\overline{1|}} = vq_{69} + (vp_{69})(1)$$

$u(k-1) = A_{x:\overline{y-x|}}$ is OK at $x = 69$ since

$$A_{69:\overline{1|}} = vq_{69} + (vp_{69})(1)$$

Note: While writing recursion in backward form gave us something exactly like page 119 of Bowers, in its original forward form it is comparable to problem 8.7 on page 251. Reasoning from that formula, with $\pi_h = 0$ and $b_{h+1} = 1$, should also lead to the correct answer.

Question #9
Key: A

You arrive first if both (A) the first train to arrive is a local and (B) no express arrives in the 12 minutes after the local arrives.
$P(A) = 0.75$
Expresses arrive at Poisson rate of $(0.25)(20) = 5$ per hour, hence 1 per 12 minutes.

$$f(0) = \frac{e^{-1}1^0}{0!} = 0.368$$

A and B are independent, so
$P(A \text{ and } B) = (0.75)(0.368) = 0.276$

Question #10

Key: E

$d = 0.05 \rightarrow v = 0.095$

At issue

$$A_{40} = \sum_{k=0}^{49} v^{k+1}\,{}_{k|}q_{40} = 0.02\left(v^1 + ... + v^{50}\right) = 0.02v\left(1 - v^{50}\right)/d = 0.35076$$

and $\ddot{a}_{40} = \left(1 - A_{40}\right)/d = \left(1 - 0.35076\right)/0.05 = 12.9848$

so $P_{40} = \dfrac{1000A_{40}}{\ddot{a}_{40}} = \dfrac{350.76}{12.9848} = 27.013$

$$E\left({}_{10}L \,\middle|\, K(40) \geq 10\right) = 1000A_{50}^{\text{Revised}} - P_{40}\ddot{a}_{50}^{\text{Revised}} = 549.18 - (27.013)(9.0164) = 305.62$$

where

$$A_{50}^{\text{Revised}} = \sum_{k=0}^{24} v^{k+1}\,{}_{k|}q_{50}^{\text{Revised}} = 0.04\left(v^1 + ... + v^{25}\right) = 0.04v\left(1 - v^{25}\right)/d = 0.54918$$

and $\ddot{a}_{50}^{\text{Revised}} = \left(1 - A_{50}^{\text{Revised}}\right)/d = \left(1 - 0.54918\right)/0.05 = 9.0164$

Question #11
Key: E

Let NS denote non-smokers and S denote smokers.

The shortest solution is based on the conditional variance formula

$$\operatorname{Var}(X) = E(\operatorname{Var}(X|Y)) + \operatorname{Var}(E(X|Y))$$

Let Y = 1 if smoker; Y = 0 if non-smoker

$$E\left(\bar{a}_{\overline{T}|}|Y=1\right) = \bar{a}_x^S = \frac{1-\bar{A}_x^S}{\delta}$$

$$= \frac{1-0.444}{0.1} = 5.56$$

Similarly $E\left(\bar{a}_{\overline{T}|}|Y=0\right) = \frac{1-0.286}{0.1} = 7.14$

$$E\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right) = E\left(E\left(\bar{a}_{\overline{T}|}|0\right)\right)\times \text{Prob}(\text{Y=0}) + E\left(E\left(\bar{a}_{\overline{T}|}|1\right)\right)\times \text{Prob}(\text{Y=1})$$

$$= (7.14)(0.70) + (5.56)(0.30)$$

$$= 6.67$$

$$E\left[\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right)^2\right] = \left(7.14^2\right)(0.70) + \left(5.56^2\right)(0.30)$$

$$= 44.96$$

$$\operatorname{Var}\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right) = 44.96 - 6.67^2 = 0.47$$

$$E\left(\operatorname{Var}\left(\bar{a}_{\overline{T}|}|Y\right)\right) = (8.503)(0.70) + (8.818)(0.30)$$

$$= 8.60$$

$$\operatorname{Var}\left(\bar{a}_{\overline{T}|}\right) = 8.60 + 0.47 = 9.07$$

Alternatively, here is a solution based on

$\operatorname{Var}(Y) = E\left(Y^2\right) - \left[E(Y)\right]^2$, a formula for the variance of any random variable. This can be

transformed into $E\left(Y^2\right) = \operatorname{Var}(Y) + \left[E(Y)\right]^2$ which we will use in its conditional form

$$E\left(\left(\bar{a}_{\overline{T}|}\right)^2|\text{NS}\right) = \operatorname{Var}\left(\bar{a}_{\overline{T}|}|\text{NS}\right) + \left[E\left(\bar{a}_{\overline{T}|}|\text{NS}\right)\right]^2$$

$$\operatorname{Var}\left[\bar{a}_{\overline{T}|}\right] = E\left[\left(\bar{a}_{\overline{T}|}\right)^2\right] - \left(E\left[\bar{a}_{\overline{T}|}\right]\right)^2$$

$$E\left[\bar{a}_{\overline{T}|}\right] = E\left[\bar{a}_{\overline{T}|}|\text{S}\right]\times \text{Prob}[\text{S}] + E\left[\bar{a}_{\overline{T}|}|\text{NS}\right]\times \text{Prob}[\text{NS}]$$

$$= 0.30\bar{a}_x^{\text{S}} + 0.70\bar{a}_x^{\text{NS}}$$

$$= \frac{0.30\left(1-\bar{A}_x^{\text{S}}\right)}{0.1} + \frac{0.70\left(1-\bar{A}_x^{\text{NS}}\right)}{0.1}$$

$$= \frac{0.30(1-0.444)+0.70(1-0.286)}{0.1} = (0.30)(5.56)+(0.70)(7.14)$$

$$= 1.67 + 5.00 = 6.67$$

$$E\left[\left(\bar{a}_{\overline{T}|}\right)^2\right] = E\left[\bar{a}_{\overline{T}|}^{\,2} \middle| \text{S}\right] \times \text{Prob}[\text{S}] + E\left[\bar{a}_{\overline{T}|}^{\,2} \middle| \text{NS}\right] \times \text{Prob}[\text{NS}]$$

$$= 0.30\left(\text{Var}\left(\bar{a}_{\overline{T}|} \middle| \text{S}\right) + \left(E\left[\bar{a}_{\overline{T}|} \middle| \text{S}\right]\right)^2\right)$$

$$+0.70\left(\text{Var}\left(\bar{a}_{\overline{T}|} \middle| \text{NS}\right) + E\left(\bar{a}_{\overline{T}|} \middle| \text{NS}\right)^2\right)$$

$$= 0.30\left[8.818 + (5.56)^2\right] + 0.70\left[8.503 + (7.14)^2\right]$$

11.919 + 41.638 = 53.557

$$\text{Var}\left[\bar{a}_{\overline{T}|}\right] = 53.557 - (6.67)^2 = 9.1$$

Alternatively, here is a solution based on $\bar{a}_{\overline{T}|} = \frac{1-v^T}{\delta}$

$$\text{Var}\left(\bar{a}_{\overline{T}|}\right) = \text{Var}\left(\frac{1}{\delta} - \frac{v^T}{\delta}\right)$$

$$= \text{Var}\left(\frac{-v^T}{\delta}\right) \text{ since } \text{Var}(X + \text{constant}) = \text{Var}(X)$$

$$= \frac{\text{Var}\left(v^T\right)}{\delta^2} \text{ since } \text{Var}(\text{constant} \times X) = \text{constant}^2 \times \text{Var}(X)$$

$$= \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{\delta^2} \text{ which is Bowers formula 5.2.9}$$

This could be transformed into ${}^2A_x = \delta^2 \,\text{Var}\left(\bar{a}_{\overline{T}|}\right) + \bar{A}_x^2$, which we will use to get ${}^2A_x^{\text{NS}}$ and ${}^2A_x^{\text{S}}$.

$$
\begin{aligned}
{}^2A_x &= E\left[v^{2T}\right] \\
&= E\left[v^{2T}\,|\text{NS}\right]\times \text{Prob}(\text{NS}) + E\left[v^{2T}\,|\text{S}\right]\times \text{Prob}(\text{S}) \\
&= \left[\delta^2 \text{Var}\left(\bar{a}_{\overline{T|}}\,|\text{NS}\right) + \left(\bar{A}_x^{\text{NS}}\right)^2\right]\times \text{Prob}(\text{NS}) \\
&+ \left[\delta^2 \text{Var}\left(\bar{a}_{\overline{T|}}\,|\text{S}\right) + \left(\bar{A}_x^{\text{S}}\right)^2\right]\times \text{Prob}(\text{S}) \\
&= \left[(0.01)(8.503) + 0.286^2\right]\times 0.70 \\
&+ \left[(0.01)(8.818) + 0.444^2\right]\times 0.30 \\
&= (0.16683)(0.70) + (0.28532)(0.30) \\
&= 0.20238
\end{aligned}
$$

$$
\begin{aligned}
\bar{A}_x &= E\left[v^{T}\right] \\
&= E\left[v^{T}\,|\text{NS}\right]\times \text{Prob}(\text{NS}) + E\left[v^{T}\,|\text{S}\right]\times \text{Prob}(\text{S}) \\
&= (0.286)(0.70) + (0.444)(0.30) \\
&= 0.3334
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}\left(\bar{a}_{\overline{T|}}\right) &= \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{\delta^2} \\
&= \frac{0.20238 - 0.3334^2}{0.01} = 9.12
\end{aligned}
$$

Question #12
Key: A

To be a density function, the integral of *f* must be 1 (i.e., everyone dies eventually). The solution is written for the general case, with upper limit ∞. Given the distribution of $f_2(t)$, we could have used upper limit 100 here.

Preliminary calculations from the Illustrative Life Table:

$$
\frac{l_{50}}{l_0} = 0.8951
$$

$$
\frac{l_{40}}{l_0} = 0.9313
$$

$$1 = \int_0^\infty f_T(t)\,dt = \int_0^{50} k\, f_1(t)\,dt + \int_{50}^\infty 1.2\, f_2(t)dt$$
$$= k\int_0^{50} f_1(t)\,dt + 1.2\int_{50}^\infty f_2(t)dt$$
$$= k F_1(50) + 1.2\left(F_2(\infty) - F_2(50)\right)$$
$$= k\left(1 - {}_{50}p_0\right) + 1.2(1 - 0.5)$$
$$= k(1 - 0.8951) + 0.6$$
$$k = \frac{1-0.6}{1-0.8951} = 3.813$$

For $x \le 50, F_T(x) = \int_0^x 3.813 f_1(t)\,dt = 3.813 F_1(x)$

$$F_T(40) = 3.813\left(1 - \frac{l_{40}}{l_0}\right) = 3.813(1 - 0.9313) = 0.262$$
$$F_T(50) = 3.813\left(1 - \frac{l_{50}}{l_0}\right) = 3.813(1 - 0.8951) = 0.400$$
$${}_{10}p_{40} = \frac{1 - F_T(50)}{1 - F_T(40)} = \frac{1 - 0.400}{1 - 0.262} = 0.813$$

Question #13
Key: D

Let NS denote non-smokers, S denote smokers.

$$\text{Prob}(T < t) = \text{Prob}(T < t|\text{NS}) \times \text{Prob}(\text{NS}) + \text{Prob}(T < t|\text{S}) \times \text{Prob}(\text{S})$$
$$= \left(1 - e^{-0.1t}\right) \times 0.7 + \left(1 - e^{-0.2t}\right) \times 0.3$$
$$= 1 - 0.7e^{-0.1t} - 0.3e^{-0.2t}$$

$S(t) = 0.3e^{-0.2t} + 0.7e^{-0.1t}$

Want $\hat{t}$ such that $0.75 = 1 - S(\hat{t})$ or $0.25 = S(\hat{t})$

$$0.25 = 0.3e^{-2\hat{t}} + 0.7e^{-0.1\hat{t}} = 0.3\left(e^{-0.1\hat{t}}\right)^2 + 0.7e^{-0.1\hat{t}}$$

Substitute: let $x = e^{-0.1\hat{t}}$

$0.3x^2 + 0.7x - 0.25 = 0$

This is quadratic, so $x = \frac{-0.7 \pm \sqrt{0.49 + (0.3)(0.25)4}}{2(0.3)}$

$x = 0.3147$

$e^{-0.1\hat{t}} = 0.3147 \quad \text{so } \hat{t} = 11.56$

Question #14
Key: D

The modified severity, X*, represents the conditional payment amount given that a payment occurs. Given that a payment is required (X > *d*), the payment must be uniformly distributed between 0 and $c \cdot (b-d)$.

The modified frequency, N*, represents the number of losses that result in a payment. The deductible eliminates payments for losses below *d*, so only $1 - F_x(d) = \frac{b-d}{b}$ of losses will require payments. Therefore, the Poisson parameter for the modified frequency distribution is $\lambda \cdot \frac{b-d}{b}$. (Reimbursing c% after the deductible affects only the payment amount and not the frequency of payments).

Question #15
Key: C

Let *N* = number of sales on that day
S = aggregate prospective loss at issue on those sales
K = curtate future lifetime

$$N \sim \text{Poisson}(0.2 * 50) \qquad \Rightarrow E[N] = \text{Var}[N] = 10$$

$$_0L = 10{,}000 v^{K+1} - 500 \ddot{a}_{\overline{K+1}|} \qquad \Rightarrow E[_0L] = 10{,}000 A_{65} - 500 \ddot{a}_{65}$$

$$_0L = \left(10{,}000 + \frac{500}{d}\right) v^{K+1} - \frac{500}{d} \qquad \Rightarrow Var[_0L] = \left(10{,}000 + \frac{500}{d}\right)^2 \left[{}^2A_{65} - (A_{65})^2 \right]$$

$$S = {}_0L_1 + {}_0L_2 + \ldots + {}_0L_N$$

$$E[S] = E[N] \cdot E[_0L]$$

$$Var[S] = Var[_0L] \cdot E[N] + (E[_0L])^2 \cdot \text{Var}[N]$$

$$\Pr(S<0)=\Pr\left(Z<\frac{0-E[S]}{\sqrt{Var[S]}}\right)$$

Substituting d = 0.06/(1+0.06), ${}^2A_{65}=0.23603$, $A_{65}=0.43980$ and $\ddot{a}_{65}=9.8969$ yields

$$E[{}_0L]=-550.45$$

$$\text{Var}[{}_0L]=15{,}112{,}000$$

$$E[S]=-5504.5$$

$$\text{Var}[S]=154{,}150{,}000$$

Std Dev (S) = 12,416

$$\Pr(S<0)=\Pr\left(\frac{S+5504.5}{12{,}416}<\frac{5504.5}{12{,}416}\right)$$

$$=\Pr(Z<0.443)$$

$$=0.67$$

With the answer choices, it was sufficient to recognize that:

$$0.6554=\phi(0.4)<\phi(0.443)<\phi(0.5)=0.6915$$

By interpolation, $\phi(0.443)\approx(0.43)\phi(0.5)+(0.57)\phi(0.4)$

$$=(0.43)(0.6915)+(0.57)(0.6554)$$

$$=0.6709$$

Question #16
Key: A

$$1000P_{40} = \frac{A_{40}}{\ddot{a}_{40}} = \frac{161.32}{14.8166} = 10.89$$

$$1000\,{}_{20}V_{40} = 1000\left(1 - \frac{\ddot{a}_{60}}{\ddot{a}_{40}}\right) = 1000\left(1 - \frac{11.1454}{14.8166}\right) = 247.78$$

$${}_{21}V = \frac{\left({}_{20}V + 5000P_{40}\right)(1+i) - 5000q_{60}}{P_{60}}$$

$$= \frac{\left(247.78 + (5)(10.89)\right)\times 1.06 - 5000\left(0.01376\right)}{1 - 0.01376} = 255$$

[Note: For this insurance, ${}_{20}V = 1000\,{}_{20}V_{40}$ because retrospectively, this is identical to whole life]

Though it would have taken <u>much</u> longer, you can do this as a prospective reserve. The prospective solution is included for educational purposes, not to suggest it would be suitable under exam time constraints.

$1000P_{40} = 10.89$ as above

$$1000A_{40} + 4000\,{}_{20}E_{40}A^{1}_{60:\overline{5}|} = 1000P_{40} + 5000P_{40} \times {}_{20}E_{40}\,\ddot{a}_{60:\overline{5}|} + \pi\,{}_{20}E_{40} \times {}_{5}E_{60}\,\ddot{a}_{65}$$

where $A^{1}_{60:\overline{5}|} = A_{60} - {}_{5}E_{60}\,A_{65} = 0.06674$

$$\ddot{a}_{40:\overline{20}|} = \ddot{a}_{40} - {}_{20}E_{40}\,\ddot{a}_{60} = 11.7612$$

$$\ddot{a}_{60:\overline{5}|} = \ddot{a}_{60} - {}_{5}E_{60}\,\ddot{a}_{65} = 4.3407$$

$$1000\left(0.16132\right) + \left(4000\right)\left(0.27414\right)\left(0.06674\right) =$$

$$= \left(10.89\right)\left(11.7612\right) + \left(5\right)\left(10.89\right)\left(0.27414\right)\left(4.3407\right) + \pi\left(0.27414\right)\left(0.68756\right)\left(9.8969\right)$$

$$\pi = \frac{161.32 + 73.18 - 128.08 - 64.79}{1.86544}$$

$$= 22.32$$

Having struggled to solve for π, you could calculate ${}_{20}V$ prospectively then (as above)
calculate ${}_{21}V$ recursively.

$${}_{20}V = 4000A^{1}_{60:\overline{5}|} + 1000A_{60} - 5000P_{40}\,\ddot{a}_{60:\overline{5}|} - \pi\,{}_{5}E_{60}\,\ddot{a}_{65}$$

$$= \left(4000\right)\left(0.06674\right) + 369.13 - \left(5000\right)\left(0.01089\right)\left(4.3407\right) - \left(22.32\right)\left(0.68756\right)\left(9.8969\right)$$

$$= 247.86 \text{ (minor rounding difference from } 1000\,{}_{20}V_{40})$$

Or we can continue to ${}_{21}V$ prospectively

$${}_{21}V = 5000A^{1}_{61:\overline{4}|} + 1000\,{}_{4}E_{61}\,A_{65} - 5000P_{40}\,\ddot{a}_{61:\overline{4}|} - \pi\,{}_{4}E_{61}\,\ddot{a}_{65}$$

where ${}_{4}E_{61} = \dfrac{l_{65}}{l_{61}}v^4 = \left(\dfrac{7,533,964}{8,075,403}\right)(0.79209) = 0.73898$

$$A^{1}_{61:\overline{4}|} = A_{61} - {}_{4}E_{61}\,A_{65} = 0.38279 - 0.73898 \times 0.43980$$
$$= 0.05779$$
$$\ddot{a}_{61:\overline{4}|} = \ddot{a}_{61} - {}_{4}E_{61}\,\ddot{a}_{65} = 10.9041 - 0.73898 \times 9.8969$$
$$= 3.5905$$

$${}_{21}V = (5000)(0.05779) + (1000)(0.73898)(0.43980)$$
$$-(5)(10.89)(3.5905) - 22.32(0.73898)(9.8969)$$
$$= 255$$

Finally. A moral victory. Under exam conditions since prospective benefit reserves must equal retrospective benefit reserves, calculate whichever is simpler.

Question #17
Key: C

$$\operatorname{Var}(Z) = {}^{2}A_{41} - (A_{41})^2$$
$$A_{41} - A_{40} = 0.00822 = A_{41} - (vq_{40} + vp_{40}A_{41})$$
$$= A_{41} - (0.0028/1.05 + (0.9972/1.05)A_{41})$$
$$\Rightarrow A_{41} = 0.21650$$

$${}^{2}A_{41} - {}^{2}A_{40} = 0.00433 = {}^{2}A_{41} - \left(v^2 q_{40} + v^2 p_{40}\,{}^{2}A_{41}\right)$$
$$= {}^{2}A_{41} - (0.0028/1.05^2 + \left(0.9972/1.05^2\right)^2 A_{41})$$
$${}^{2}A_{41} = 0.07193$$

$$\operatorname{Var}(Z) = 0.07193 - 0.21650^2$$
$$= 0.02544$$

Question #18
Key: D

This solution looks imposing because there is no standard notation. Try to focus on the big picture ideas rather than starting with the details of the formulas.

Big picture ideas:

1. We can express the present values of the perpetuity recursively.
2. Because the interest rates follow a Markov process, the present value (at time t) of the future payments at time t depends only on the state you are in at time t, not how you got there.
3. Because the interest rates follow a Markov process, the present value of the future payments at times t_1 and t_2 are equal if you are in the same state at times t_1 and t_2.

Method 1: Attack <u>without</u> considering the special characteristics of this transition matrix.

Let $s_k =$ state you are in at time $k \left(\text{thus } s_k = 0, 1 \text{ or } 2\right)$

Let $Y_k =$ present value, at time k, of the future payments.
Y_k is a random variable because its value depends on the pattern of discount factors, which are random. The expected value of Y_k is not constant; it depends on what state we are in at time *k*.

Recursively we can write

$Y_k = v \times \left(1 + Y_{k+1}\right)$, where it would be better to have notation that indicates the *v*'s are not constant, but are realizations of a random variable, where the random variable itself has different distributions depending on what state we're in. However, that would make the notation so complex as to mask the simplicity of the relationship.

Every time we are in state 0 we have

$$E\left[Y_k \middle| s_k = 0\right] = 0.95 \times \left(1 + E\left[Y_{k+1} \middle| s_k = 0\right]\right)$$

$$= 0.95 \times \left(1 + \left\{\left(E\left[Y_{k+1} \middle| s_{k+1} = 0\right]\right)\right) \times \text{Prob}\left(\text{s}_{\text{k+1}} = 0 \middle| s_k = 0\right]\right)$$

$$+ \left(E\left[Y_{k+1} \middle| s_{k+1} = 1\right]\right) \times \text{Prob}\left(s_{k+1} = 1 \middle| s_k = 0\right]\right)$$

$$= \left(E\left[Y_{k+1} \middle| s_{k+1} = 2\right]\right) \times \text{Prob}\left(s_{k+1} = 2 \middle| s_k = 0\right]\right)\}$$

$$= 0.95 \times \left(1 + E\left[Y_{k+1} \middle| s_{k+1} = 1\right]\right)$$

That last step follows because from the transition matrix if we are in state 0, we always move to state 1 one period later.

Similarly, every time we are in state 2 we have

$$E\left[Y_k \middle| s_k = 2\right] = 0.93 \times \left(1 + E\left[Y_{k+1} \middle| s_k = 2\right]\right)$$

$$= 0.93 \times \left(1 + E\left[Y_{k+1} \middle| s_{k+1} = 1\right]\right)$$

That last step follows because from the transition matrix if we are in state 2, we always move to state 1 one period later.

Finally, every time we are in state 1 we have

$$E\left[Y_k \middle| s_k = 1\right] = 0.94 \times \left(1 + E\left[Y_{k+1} \middle| s_k = 1\right]\right)$$

$$= 0.94 \times \left(1 + \left\{E\left[Y_{k+1} \middle| s_{k+1} = 0\right] \times \Pr\left[s_{k+1} = 0 \middle| s_k = 1\right] + E\left[Y_{k+1} \middle| s_{k+1} = 2\right] \times \Pr\left[s_{k+1} = 2 \middle| s_k = 1\right]\right\}\right)$$

$$= 0.94 \times \left(1 + \left\{E\left[Y_{k+1} \middle| s_{k+1} = 0\right] \times 0.9 + E\left[Y_{k+1} \middle| s_{k+1} = 2\right] \times 0.1\right\}\right)$$

. Those last two steps follow from the fact that from state 1 we always go to either state 0 (with probability 0.9) or state 2 (with probability 0.1).

Now let's write those last three paragraphs using this shorter notation: $x_n = E\left[Y_k \middle| s_k = n\right]$. We can do this because (big picture idea #3), the conditional expected value is only a function of the state we are in, not when we are in it or how we got there.

$$x_0 = 0.95\left(1 + x_1\right)$$

$$x_1 = 0.94\left(1 + 0.9x_0 + 0.1x_2\right)$$

$$x_2 = 0.93\left(1 + x_1\right)$$

That's three equations in three unknowns. Solve (by substituting the first and third into the second) to get $x_1 = 16.82$.

That's the answer to the question, the expected present value of the future payments given in state 1.

The solution above is almost exactly what we would have to do with <u>any</u> 3×3 transition matrix. As we worked through, we put only the non-zero entries into our formulas. But if for example the top row of the transition matrix had been $\begin{pmatrix} 0.4 & 0.5 & 0.1 \end{pmatrix}$, then the first of our three equations would have become $x_0 = 0.95\left(1 + 0.4x_0 + 0.5x_1 + 0.1x_2\right)$, similar in structure to our actual equation for x_1. We would still have ended up with three linear equations in three unknowns, just more tedious ones to solve.

Method 2: Recognize the patterns of changes for <u>this particular transition matrix</u>.

This particular transition matrix has a recurring pattern that leads to a much quicker solution. We are starting in state 1 and are guaranteed to be back in state 1 two steps later, with the same prospective value then as we have now.
Thus,

$$E[Y] = E[Y | \text{first move is to } 0] \times \Pr[\text{first move is to } 0] + E[Y | \text{first move is to } 2] \times \Pr[\text{first move is to } 2]$$

$$= 0.94 \times \left[\left(1 + 0.95 \times \left(1 + E[Y]\right)\right)\right] \times 0.9 + \left[0.94 \times \left[\left(1 + 0.93 \times \left(1 + E[Y]\right)\right) \times 0.1\right]\right]$$

(Note that the equation above is exactly what you get when you substitute x_0 and x_2 into the formula for x_1 in Method 1.)

$$= 1.6497 + 0.8037 E[Y] + 0.1814 + 0.0874 E[Y]$$

$$E[Y] = \frac{1.6497 + 0.1814}{(1 - 0.8037 - 0.0874)}$$

$$= 16.82$$

Question #19
Key: E

The number of problems solved in 10 minutes is Poisson with mean 2.
If she solves exactly one, there is 1/3 probability that it is #3.
If she solves exactly two, there is a 2/3 probability that she solved #3.
If she solves #3 or more, she got #3.

f(0) = 0.1353
f(1) = 0.2707
f(2) = 0.2707

$$P = \left(\frac{1}{3}\right)(0.2707) + \left(\frac{2}{3}\right)(0.2707) + (1 - 0.1353 - 0.2707 - 0.2707) = 0.594$$

Question #20
Key: D

$$\mu_x^{(\tau)} = \mu_x^{(1)}(t) + \mu_x^{(2)}(t)$$

$$= 0.2\mu_x^{(\tau)}(t) + \mu_x^{(2)}(t)$$

$$\Rightarrow \mu_x^{(2)}(t) = 0.8\mu_x^{(\tau)}(t)$$

$$q_x'^{(1)} = 1 - p_x'^{(1)} = 1 - e^{-\int_0^1 0.2k\,t^2\,dt} = 1 - e^{-0.2\frac{k}{3}} = 0.04$$

$$k/3 \Rightarrow \ln(1-0.04)/(-0.2) = 0.2041$$
$$k = 0.6123$$

$$_2q_x^{(2)} = \int_0^2 {}_tp_x^{(\tau)}\mu_x^{(2)}\;dt = 0.8\int_0^2 {}_tp_x^{(\tau)}\;\mu_x^{(\tau)}(t)\;dt$$

$$= 0.8\;{}_2q_x^{(\tau)} = 0.8\left(1 - {}_2p_x^{(\tau)}\right)$$
$$_2p_x^{(\tau)} = e^{-\int_0^2 \mu_x(t)dt}$$
$$= e^{-\int_0^2 kt^2\,dt}$$
$$= e^{\frac{-8k}{3}}$$
$$= e^{\frac{-(8)(0.6123)}{3}}$$
$$= 0.19538$$

$$_2q_x^{(2)} = 0.8(1-0.19538) = 0.644$$

Question #21
Key: A

k	$k \wedge 3$	$f(k)$	$f(k)\times(k\wedge 3)$	$f(k)\times(k\wedge 3)^2$
0	0	0.1	0	0
1	1	(0.9)(0.2) = 0.18	0.18	0.18
2	2	(0.72)(0.3) = 0.216	0.432	0.864
3+	3	1-0.1-0.18-0.216 = 0.504	1.512	4.536
			2.124	5.580

$E(K \wedge 3) = 2.124$

$E\left((K \wedge 3)^2\right) = 5.580$

Var$(K \wedge 3) = 5.580 - 2.124^2 = 1.07$

Note that $E[K \wedge 3]$ is the temporary curtate life expectancy, $e_{x:\overline{3}|}$ if the life is age *x*. Problem 3.17 in Bowers, pages 86 and 87, gives an alternative formula for the variance, basing the calculation on ${}_k p_x$ rather than ${}_{k|}q_x$.

Question #22
Key: E

$$\begin{aligned} f(x) &= 0.01, \quad 0 \le x \le 80 \\ &= 0.01 - 0.00025(x-80) = 0.03 - 0.00025x, \quad 80 < x \le 120 \end{aligned}$$

$$\begin{aligned} E(x) &= \int_0^{80} 0.01x\,dx + \int_{80}^{120}\left(0.03x - 0.00025x^2\right)dx \\ &= \frac{0.01x^2}{2}\Big|_0^{80} + \frac{0.03x^2}{2}\Big|_{80}^{120} - \frac{0.00025x^3}{3}\Big|_{80}^{120} \\ &= 32 + 120 - 101.33 = 50.66667 \end{aligned}$$

$$\begin{aligned} E(X-20)_+ &= E(X) - \int_0^{20} x f(x)\,dx - 20(1 - \int_0^{20} f(x)dx) \\ &= 50.6667 - \frac{0.01x^2}{2}\Big|_0^{20} - 20\left(1 - 0.01x\Big|_0^{20}\right) \\ &= 50.6667 - 2 - 20(0.8) = 32.6667 \end{aligned}$$

Loss Elimination Ratio = $1 - \dfrac{32.6667}{50.6667} = 0.3553$

Question #23
Key: D

Let q_{64} for Michel equal the standard q_{64} plus *c*. We need to solve for *c*.
Recursion formula for a standard insurance:

$$_{20}V_{45} = \left({}_{19}V_{45} + P_{45}\right)(1.03) - q_{64}\left(1 - {}_{20}V_{45}\right)$$

Recursion formula for Michel's insurance

$$_{20}V_{45} = \left({}_{19}V_{45} + P_{45} + 0.01\right)(1.03) - \left(q_{64} + c\right)(1 - {}_{20}V_{45})$$

The values of $_{19}V_{45}$ and $_{20}V_{45}$ are the same in the two equations because we are told Michel's benefit reserves are the same as for a standard insurance.

Subtract the second equation from the first to get:

$$\begin{aligned}
0 &= -(1.03)(0.01) + c(1 - {}_{20}V_{45}) \\
c &= \frac{(1.03)(0.01)}{\left(1 - {}_{20}V_{45}\right)} \\
&= \frac{0.0103}{1 - 0.427} \\
&= 0.018
\end{aligned}$$

Question #24
Key: B

K is the curtate future lifetime for one insured.
L is the loss random variable for one insurance.
L_{AGG} is the aggregate loss random variables for the individual insurances.
σ_{AGG} is the standard deviation of L_{AGG}.
M is the number of policies.

$$L = v^{K+1} - \pi \ddot{a}_{\overline{K+1}|} = \left(1 + \frac{\pi}{d}\right) v^{K+1} - \pi/d$$

$$\begin{aligned}
E[L] &= \left(A_x - \pi \ddot{a}_x\right) = A_x - \pi \frac{\left(1 - A_x\right)}{d} \\
&= 0.24905 - 0.025\left(\frac{0.75095}{0.056604}\right) = -0.082618
\end{aligned}$$

$$\text{Var}[L] = \left(1+\frac{\pi}{d}\right)^2 \left({}^2A_x - A_x^2\right) = \left(1+\frac{0.025}{0.056604}\right)^2 \left(0.09476 - (0.24905)^2\right) = 0.068034$$

$$E[L_{AGG}] = M\,E[L] = -0.082618M$$

$$\text{Var}[L_{AGG}] = M\ \text{Var}[L] = M(0.068034) \Rightarrow \sigma_{AGG} = 0.260833\sqrt{M}$$

$$\Pr[L_{AGG} > 0] = \left[\frac{L_{AGG} - E[L_{AGG}]}{\sigma_{AGG}} > \frac{-E(L_{AGG})}{\sigma_{AGG}}\right]$$

$$\approx \Pr\left(N(0,1) > \frac{0.082618M}{\sqrt{M}\,(0.260833)}\right)$$

$$\Rightarrow 1.645 = \frac{0.082618\sqrt{M}}{0.260833}$$

$$\Rightarrow M = 26.97$$

$\Rightarrow$ minimum number needed = 27

Question #25
Key: D

Annuity benefit: $Z_1 = 12{,}000\dfrac{1-v^{K+1}}{d}$ for $K = 0,1,2,\ldots$

Death benefit: $Z_2 = Bv^{K+1}$ for $K = 0,1,2,\ldots$

New benefit:

$$Z = Z_1 + Z_2 = 12{,}000\frac{1-v^{K+1}}{d} + Bv^{K+1}$$

$$= \frac{12{,}000}{d} + \left(B - \frac{12{,}000}{d}\right)v^{K+1}$$

$$\text{Var}(Z) = \left(B - \frac{12{,}000}{d}\right)^2 \text{Var}\left(v^{K+1}\right)$$

$$\text{Var}(Z) = 0 \text{ if } B = \frac{12{,}000}{0.08} = 150{,}000\,.$$

In the first formula for $\text{Var}(Z)$, we used the formula, valid for any constants *a* and *b* and random variable *X*,

$$\text{Var}(a+bX) = b^2\text{Var}(X)$$

Question #26
Key: B

First restate the table to be CAC's cost, after the 10% payment by the auto owner:

Towing Cost, *x*	*p*(*x*)
72	50%
90	40%
144	10%

Then $E(X) = 0.5*72 + 0.4*90 + 0.1*144 = 86.4$

$E(X^2) = 0.5*72^2 + 0.4*90^2 + 0.1*144^2 = 7905.6$

$\text{Var}(X) = 7905.6 - 86.4^2 = 440.64$

Because Poisson, $E(N) = \text{Var}(N) = 1000$

$E(S) = E(X)E(N) = 86.4*1000 = 86,400$

$\text{Var}(S) = E(N)\text{Var}(X) + E(X)^2 \text{Var}(N) = 1000*440.64 + 86.4^2*1000 = 7,905,600$

$$\Pr(S > 90,000) + \Pr\left(\frac{S - E(S)}{\sqrt{\text{Var}(S)}} > \frac{90,000 - 86,400}{\sqrt{7,905,600}}\right) = \Pr(Z > 1.28) = 1 - \Phi(1.28) = 0.10$$

Since the frequency is Poisson, you could also have used

$\text{Var}(S) = \lambda E(X^2) = (1000)(7905.6) = 7,905,600$

That way, you would not need to have calculated $\text{Var}(X)$.

Question #27

Key: C

$$\text{LER} = \frac{E(X \wedge d)}{E(X)} = \frac{\theta\left(1 - e^{-d/\theta}\right)}{\theta} = 1 - e^{-d/\theta}$$

Last year $0.70 = 1 - e^{-d/\theta} \Rightarrow -d = \theta \log\ 0.30$

Next year: $-d_{new} = \theta \log(1 - \text{LER}_{new})$

Hence $\theta \log\left(1 - \text{LER}_{\text{new}}\right) = -d_{\text{new}} = \frac{4}{3}\theta \log\ 0.30$

$\log\left(1 - \text{LER}_{new}\right) = -1.6053$

$\left(1 - \text{LER}_{new}\right) = e^{-1.6053} = 0.20$

$\text{LER}_{new} = 0.80$

Question #28
Key: E

$$E(X)=e(d)S(d)+E(X\wedge d)$$ [Klugman Study Note, formula 3.10]

$$62=\mathring{e}_{40}\times {}_{40}p_0+E(T\wedge 40)$$

$$62=(\mathring{e}_{40})(0.6)+40-(0.005)(40^2)$$

$$=0.6\mathring{e}_{40}+32$$

$$\mathring{e}_{40}=\frac{(62-32)}{0.6}=50$$

The first equation, in the notation of Bowers, is $\mathring{e}_0=\mathring{e}_{40}\times {}_{40}p_0+\mathring{e}_{0:\overline{40}|}$. The corresponding formula, with $i>0$, is a very commonly used one:

$$\bar{a}_x=\bar{a}_{x:\overline{n}|}+{}_nE_x\,\bar{a}_{x+n}$$

Question #29
Key: B

$$d=0.05\Rightarrow v=0.95$$

Step 1 Determine p_x from Kevin's work:

$$608+350vp_x=1000vq_x+1000v^2p_x\left(p_{x+1}+q_{x+1}\right)$$
$$608+350(0.95)p_x=1000(0.95)(1-p_x)+1000(0.9025)p_x(1)$$
$$608+332.5p_x=950(1-p_x)+902.5p_x$$
$$p_x=342/380=0.9$$

Step 2 Calculate $1000P_{x:\overline{2}|}$, as Kira did:

$$608+350(0.95)(0.9)=1000P_{x:\overline{2}|}\left[1+(0.95)(0.9)\right]$$
$$1000P_{x:\overline{2}|}=\frac{[299.25+608]}{1.855}=489.08$$

The first line of Kira's solution is that the actuarial present value of Kevin's benefit premiums is equal to the actuarial present value of Kira's, since each must equal the actuarial present value of benefits. The actuarial present value of benefits would also have been easy to calculate as

$$(1000)(0.95)(0.1)+(1000)\left(0.95^2\right)(0.9)=907.25$$

Question #30
Key: E

Because no premiums are paid after year 10 for (x), ${}_{11}V_x = A_{x+11}$

Rearranging 8.3.10 from Bowers, we get ${}_{h+1}V = \dfrac{\left({}_hV + \pi_h\right)(1+i) - b_{h+1}q_{x+h}}{p_{x+h}}$

$${}_{10}V = \frac{(32,535 + 2,078)\times(1.05) - 100,000\times 0.011}{0.989} = 35,635.642$$

$${}_{11}V = \frac{(35,635.642 + 0)\times(1.05) - 100,000\times 0.012}{0.988} = 36,657.31 = A_{x+11}$$

Question #31
Key: B

For De Moivre's law where $s(x) = \left(1 - \dfrac{x}{\omega}\right)$:

$$\overset{\circ}{e}_x = \frac{\omega - x}{2} \ and \ {}_tp_x = \left(1 - \frac{t}{\omega - x}\right)$$

$$\overset{\circ}{e}_{45} = \frac{105-45}{2} = 30$$

$$\overset{\circ}{e}_{65} = \frac{105-65}{2} = 20$$

$$\overset{\circ}{e}_{45:65} = \int_0^{40} {}_tp_{45:65}dt = \int_0^{40} \frac{60-t}{60} \times \frac{40-t}{40} dt$$

$$= \frac{1}{60\times 40}\left(60\times 40\times t - \frac{60+40}{2}t^2 + \frac{1}{3}t^3\right)\Bigg|_0^{40}$$

$$= 15.56$$

$$\overset{\circ}{e}_{\overline{45:65}} = \overset{\circ}{e}_{45} + \overset{\circ}{e}_{65} - \overset{\circ}{e}_{45:65}$$

$$= 30 + 20 - 15.56 = 34$$

In the integral for $\overset{\circ}{e}_{45:65}$, the upper limit is 40 since 65 (and thus the joint status also) can survive a maximum of 40 years.

Question #32
Answer: E

$$\mu(4) = -s'(4)/s(4)$$

$$= \frac{-\left(-e^4/100\right)}{1-e^4/100}$$

$$= \frac{e^4/100}{1-e^4/100}$$

$$= \frac{e^4}{100-e^4}$$

$$= 1.202553$$

Question # 33
Answer: A

$$q_x^{(i)} = q_x^{(\tau)}\left[\frac{\ln\ p_x'^{(i)}}{\ln\ p_x^{(\tau)}}\right] = q_x^{(\tau)}\left[\frac{\ln\ e^{-\mu^{(i)}}}{\ln\ e^{-\mu^{(\tau)}}}\right]$$

$$= q_x^{(\tau)} \times \frac{\mu^{(i)}}{\mu^{(\tau)}}$$

$$\mu_x{}^{(\tau)} = \mu_x{}^{(1)} + \mu_x{}^{(2)} + \mu_x{}^{(3)} = 1.5$$

$$q_x^{(\tau)} = 1 - e^{-\mu(\tau)} = 1 - e^{-1.5}$$

=0.7769

$$q_x^{(2)} = \frac{(0.7769)\mu^{(2)}}{\mu^{(\tau)}} = \frac{(0.5)(0.7769)}{1.5}$$

$$= 0.2590$$

Question # 34
Answer: D

$$_{2|2}A_{[60]} = v^3 \times {}_2p_{[60]} \times q_{[60]+2} +$$

v^3	$_2p_{[60]}$	$q_{[60]+2}$
↓	↓	↓
pay at end of year 3	live 2 years	then die in year 3

$+v^4$	$_3p_{[60]}$	q_{60+3}
pay at end of year 4	live 3 years	then die in year 4

$$= \frac{1}{(1.03)^3}(1-0.09)(1-0.11)(0.13) + \frac{1}{(1.03)^4}(1-0.09)(1-0.11)(1-0.13)(0.15)$$

$$= 0.19$$

Question # 35
Answer: B

$$\bar{a}_x = \bar{a}_{x:\overline{5|}} + {}_5E_x\, \bar{a}_{x+5}$$

$\bar{a}_{x:\overline{5|}} = \dfrac{1-e^{-0.07(5)}}{0.07} = 4.219$, where $0.07 = \mu + \delta$ for $t < 5$

$$_5E_x = e^{-0.07(5)} = 0.705$$

$\bar{a}_{x+5} = \dfrac{1}{0.08} = 12.5$, where $0.08 = \mu + \delta$ for $t \geq 5$

$$\therefore \bar{a}_x = 4.219 + (0.705)(12.5) = 13.03$$

Question # 36
Answer: E

The distribution of claims (a gamma mixture of Poissons) is negative binomial.

$$E(N) = E_{\wedge}\left(E(N|\Lambda)\right) = E_{\wedge}(\Lambda) = 3$$
$$Var(N) = E_{\wedge}\left(Var(N|\Lambda)\right) + Var_{\wedge}\left(E(N|\Lambda)\right)$$
$$= E_{\wedge}(\Lambda) + Var_{\wedge}(\Lambda) = 6$$
$$r\beta = 3$$
$$r\beta(1+\beta) = 6$$
$$(1+\beta) = 6/3 = 2\ ; \quad \beta = 1$$
$$r\beta = 3$$
$$r = 3$$

$$p_0 = (1+\beta)^{-r} = 0.125$$
$$p_1 = \frac{r\beta}{(1+\beta)^{r+1}} = 0.1875$$
$$\text{Prob}(\text{at most } 1) = p_0 + p_1$$
$$= 0.3125$$

Question # 37
Answer: A

$$E(S) = E(N) \times E(X) = 110 \times 1{,}101 = 121{,}110$$

$$Var(S) = E(N) \times Var(X) + E(X)^2 \times Var(N)$$
$$= 110 \times 70^2 + 1101^2 \times 750$$
$$= 909{,}689{,}750$$

$$\text{Std Dev } (S) = 30{,}161$$

$\Pr(S < 100{,}000) = \Pr\left(Z < (100{,}000 - 121{,}110)/30{,}161\right)$ where Z has standard normal distribution

$$= \Pr(Z < -0.70) = 0.242$$

Question # 38
Answer: C

This is just the Gambler's Ruin problem, in units of 5,000 calories.
Each day, up one with $p = 0.45$; down 1 with $q = 0.55$
Will Allosaur ever be up 1 before being down 2?

$$P_2 = \frac{\left(1-(0.55/0.45)\hat{}2\right)}{\left(1-(0.55/0.45)\hat{}3\right)} = 0.598$$

Or, by general principles instead of applying a memorized formula:
Let $P_1 =$ probability of ever reaching 3 (15,000 calories) if at 1 (5,000 calories).
Let $P_2 =$ probability of ever reaching 3 (15,000 calories) if at 2 (10,000 calories).

From either, we go up with $p = 0.45$, down with $q = 0.55$

P(reaching 3) = P(up) $\times$ P(reaching 3 after up) + P(down) $\times$ P(reaching 3 after down)

$$P_2 = 0.45\times 1 + 0.55\times P_1$$
$$P_1 = 0.45\times P_2 + 0.55\times 0 = 0.45\times P_2$$
$$P_2 = 0.45 + 0.55\times P_1 = 0.45 + 0.55\times 0.45\times P_2 = 0.45 + 0.2475P_2$$
$$P_2 = 0.45/(1-0.2475) = 0.598$$

Here is another approach, feasible since the number of states is small.

Let states 0,1,2,3 correspond to 0; 5,000; 10,000; ever reached 15,000 calories. For purposes of this problem, state 3 is absorbing, since once the allosaur reaches 15,000 we don't care what happens thereafter.

The transition matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.55 & 0 & 0.45 & 0 \\ 0 & 0.55 & 0 & 0.45 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Starting with the allosaur in state 2;

[0	0	1	0]	at inception
[0	0.55	0	0.45]	after 1
[0.3025	0	0.2475	0.45]	after 2
[0.3025	0.1361	0	0.5614]	after 3
[0.3774	0	0.0612	0.5614]	after 4
[0.3774	0.0337	0	0.5889]	after 5
[0.3959	0	0.0152	0.5889]	after 6

By this step, if not before, Prob(state 3) must be converging to 0.60. It's already closer to 0.60 than 0.57, and its maximum is 0.5889 + 0.0152

Question # 39
Answer: D

Per 10 minutes, find coins worth exactly 10 at Poisson rate $(0.5)(0.2)(10)=1$

Per 10 minutes, $f(0)=0.3679$ $F(0)=0.3679$
$f(1)=0.3679$ $F(1)=0.7358$
$f(2)=0.1839$ $F(2)=0.9197$
$f(3)=0.0613$ $F(3)=0.9810$

Let Period 1 = first 10 minutes; period 2 = next 10.

Method 1, succeed with 3 or more in period 1; or exactly 2, then one or more in period 2
$$P=(1-F(2))+f(2)(1-F(0))=(1-0.9197)+(0.1839)(1-0.3679)$$
$$=0.1965$$

Method 2: fail in period 1 if < 2; $\Pr ob = F(1)=0.7358$
fail in period 2 if exactly 2 in period 1, then 0; $\Pr ob = f(2)f(0)$
$=(0.1839)(0.3679)=0.0677$
Succeed if fail neither period; $\Pr ob = 1-0.7358-0.0677$
$=0.1965$

(Method 1 is attacking the problem as a stochastic process model; method 2 attacks it as a ruin model.)

Question # 40
Answer: D

Use Mod to designate values unique to this insured.

$$\ddot{a}_{60}=(1-A_{60})/d=(1-0.36933)/\left[(0.06)/(1.06)\right]=11.1418$$

$$1000P_{60}=1000A_{60}/\ddot{a}_{60}=1000(0.36933/11.1418)=33.15$$

$$A_{60}^{Mod}=v\left(q_{60}^{Mod}+p_{60}^{Mod}A_{61}\right)=\frac{1}{1.06}\left[0.1376+(0.8624)(0.383)\right]=0.44141$$

$$\ddot{a}^{Mod} = \left(1 - A_{60}^{Mod}\right) / d = \left(1 - 0.44141\right) / \left[0.06 / 1.06\right] = 9.8684$$

$$E\left[{}_0L^{Mod}\right] = 1000\left(A_{60}^{Mod} - P_{60}\ddot{a}_{60}^{Mod}\right)$$

$$= 1000\left[0.44141 - 0.03315(9.8684)\right]$$

$$= 114.27$$

Question # 41
Answer: D

The prospective reserve at age 60 per 1 of insurance is A_{60}, since there will be no future premiums. Equating that to the retrospective reserve per 1 of coverage, we have:

$$A_{60} = P_{40}\frac{\ddot{s}_{40:\overline{10}|}}{{}_{10}E_{50}} + P_{50}^{Mod}\ \ddot{s}_{50:\overline{10}|} - {}_{20}k_{40}$$

$$A_{60} = \frac{A_{40}}{\ddot{a}_{40}} \times \frac{\ddot{a}_{40:\overline{10}|}}{{}_{10}E_{40}\ {}_{10}E_{50}} + P_{50}^{Mod}\frac{\ddot{a}_{50:\overline{10}|}}{{}_{10}E_{50}} - \frac{A^{1}_{40:\overline{20}|}}{{}_{20}E_{40}}$$

$$0.36913 = \frac{0.16132}{14.8166} \times \frac{7.70}{(0.53667)(0.51081)} + P_{50}^{Mod}\frac{7.57}{0.51081} - \frac{0.06}{0.27414}$$

$$0.36913 = 0.30582 + 14.8196\ P_{50}^{Mod} - 0.21887$$

$$1000\ P_{50}^{Mod} = 19.04$$

Alternatively, you could equate the retrospective and prospective reserves at age 50. Your equation would be:

$$A_{50} - P_{50}^{Mod}\ \ddot{a}_{50:\overline{10}|} = \frac{A_{40}}{\ddot{a}_{40}} \times \frac{\ddot{a}_{40:\overline{10}|}}{{}_{10}E_{40}} - \frac{A^{1}_{40:\overline{10}|}}{{}_{10}E_{40}}$$

where $A^{1}_{40:\overline{10}|} = A_{40} - {}_{10}E_{40}\ A_{50}$

$$= 0.16132 - (0.53667)(0.24905)$$

$$= 0.02766$$

$$0.24905-\left(P_{50}^{Mod}\right)(7.57)=\frac{0.16132}{14.8166}\times\frac{7.70}{0.53667}-\frac{0.02766}{0.53667}$$

$$1000P_{50}^{Mod}=\frac{(1000)(0.14437)}{7.57}=19.07$$

Alternatively, you could set the actuarial present value of benefits at age 40 to the actuarial present value of benefit premiums. The change at age 50 did not change the benefits, only the pattern of paying for them.

$$A_{40}=P_{40}\ \ddot{a}_{40:\overline{10}|}+P_{50}^{Mod}\ {}_{10}E_{40}\ \ddot{a}_{50:\overline{10}|}$$

$$0.16132=\left(\frac{0.16132}{14.8166}\right)(7.70)+\left(P_{50}^{Mod}\right)(0.53667)(7.57)$$

$$1000P_{50}^{Mod}=\frac{(1000)(0.07748)}{4.0626}=19.07$$

Question # 42
Answer: A

$$d_x^{(2)}=q_x^{(2)}\times l_x^{(\tau)}=400$$

$$d_x^{(1)}=0.45(400)=180$$

$$q_x'^{(2)}=\frac{d_x^{(2)}}{l_x^{(\tau)}-d_x^{(1)}}=\frac{400}{1000-180}=0.488$$

$$p_x'^{(2)}=1-0.488=0.512$$

Note: The UDD assumption was not critical except to have all deaths during the year so that 1000 - 180 lives are subject to decrement 2.

Question #43
Answer: D

Use “age” subscripts for years completed in program. E.g., p_0 applies to a person newly hired (“age” 0).

Let decrement 1 = fail, 2 = resign, 3 = other.

Then $q_0'^{(1)} = 1/4,\ q_1'^{(1)} = 1/5,\ q_2'^{(1)} = 1/3$

$q_0'^{(2)} = 1/5,\ q_1'^{(2)} = 1/3,\ q_2'^{(2)} = 1/8$

$q_0'^{(3)} = 1/10,\ q_1'^{(3)} = 1/9,\ q_2'^{(3)} = 1/4$

This gives $p_0^{(\tau)} = (1-1/4)(1-1/5)(1-1/10) = 0.54$

$p_1^{(\tau)} = (1-1/5)(1-1/3)(1-1/9) = 0.474$

$p_2^{(\tau)} = (1-1/3)(1-1/8)(1-1/4) = 0.438$

So $1_0^{(\tau)} = 200,\ 1_1^{(\tau)} = 200\,(0.54) = 108$, and $1_2^{(\tau)} = 108\,(0.474) = 51.2$

$q_2^{(1)} = \left[\log p_2'^{(1)} / \log p_2^{(\tau)}\right] q_2^{(\tau)}$

$q_2^{(1)} = \left[\log\left(\tfrac{2}{3}\right) / \log(0.438)\right]\left[1-0.438\right]$

$= (0.405/0.826)(0.562)$

$= 0.276$

$d_2^{(1)} = l_2^{(\tau)}\ q_2^{(1)}$

$= (51.2)(0.276) = 14$

Question #44
Answer: C

Let: N = number
X = profit
S = aggregate profit
subscripts G = “good”, B = “bad”, AB = “accepted bad”

$\lambda_G = \left(\tfrac{2}{3}\right)(60) = 40$

$\lambda_{AB} = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)(60) = 10$ (If you have trouble accepting this, think instead of a heads-tails rule, that the application is accepted if the applicant’s government-issued identification number, e.g. U.S. Social Security Number, is odd. It is <u>not</u> the same as saying he automatically alternates accepting and rejecting.)

$Var(S_G) = E(N_G) \times Var(X_G) + Var(N_G) \times E(X_G)^2$

$$= (40)(10{,}000) + (40)(300^2) = 4{,}000{,}000$$

$$Var(S_{AB}) = E(N_{AB}) \times Var(X_{AB}) + Var(N_{AB}) \times E(X_{AB})^2$$

$$= (10)(90{,}000) + (10)(-100)^2 = 1{,}000{,}000$$

S_G and S_{AB} are independent, so

$$Var(S) = Var(S_G) + Var(S_{AB}) = 4{,}000{,}000 + 1{,}000{,}000$$

$$= 5{,}000{,}000$$

If you don't treat it as three streams ("goods", "accepted bads", "rejected bads"), you can compute the mean and variance of the profit <u>per "bad" received</u>.

$$\lambda_B = \left(\tfrac{1}{3}\right)(60) = 20$$

If all "bads" were accepted, we would have $E(X^2{}_B) = Var(X_B) + E(X_B)^2$

$$= 90{,}000 + (-100)^2 = 100{,}000$$

Since the probability a "bad" will be accepted is only 50%,

$$E(X_B) = \text{Prob}(\text{accepted}) \times E(X_B|\text{accepted}) + \text{Prob}(\text{not accepted}) \times E(X_B|\text{not accepted})$$

$$= (0.5)(-100) + (0.5)(0) = -50$$

$$E(X_B^2) = (0.5)(100{,}000) + (0.5)(0) = 50{,}000$$

Likewise,

Now $Var(S_B) = E(N_B) \times Var(X_B) + Var(N_B) \times E(X_B)^2$

$$= (20)(47{,}500) + (20)(50^2) = 1{,}000{,}000$$

S_G and S_B are independent, so

$$Var(S) = Var(S_G) + Var(S_B) = 4{,}000{,}000 + 1{,}000{,}000$$

$$= 5{,}000{,}000$$

Question # 45
Answer: C

Let *N* = number of prescriptions then $S = N \times 40$

n	$f_N(n)$	$F_N(n)$	$1-F_N(n)$
0	0.2000	0.2000	0.8000
1	0.1600	0.3600	0.6400
2	0.1280	0.4880	0.5120
3	0.1024	0.5904	0.4096

$$E(N) = 4 = \sum_{j=0}^{\infty}\left(1-F(j)\right)$$

$$\begin{aligned}E\left[(S-80)_+\right] &= 40 \times E\left[(N-2)_+\right] = 40 \times \sum_{j=2}^{\infty}\left(1-F(j)\right) \\ &= 40 \times \left[\sum_{j=0}^{\infty}\left(1-F(j)\right) - \sum_{j=0}^{1}\left(1-F(j)\right)\right] \\ &= 40(4-1.44) = 40 \times 2.56 = 102.40\end{aligned}$$

$$\begin{aligned}E\left[(S-120)_+\right] &= 40 \times E\left[(N-3)_+\right] = 40 \times \sum_{j=3}^{\infty}\left(1-F(j)\right) \\ &= 40 \times \left[\sum_{j=0}^{\infty}\left(1-F(j)\right) - \sum_{j=0}^{2}\left(1-F(j)\right)\right] \\ &= 40(4-1.952) = 40 \times 2.048 = 81.92\end{aligned}$$

Since no values of S between 80 and 120 are possible,

$$E\left[(S-100)_+\right] = \frac{(120-100) \times E\left[(S-80)_+\right] + (100-80) \times E\left[(S-120)_+\right]}{120} = 92.16$$

Alternatively,

$$E\left[(S-100)_+\right] = \sum_{j=0}^{\infty}(40j-100) f_N(j) + 100 f_N(0) + 60 f_N(1) + 20 f_N(2)$$

(The correction terms are needed because $(40j-100)$ would be negative for j = 0, 1, 2; we need to add back the amount those terms would be negative)

$$\begin{aligned}&= 40\sum_{j=0}^{\infty} j \times f_N(j) - 100\sum_{j=0}^{\infty} f_N(j) + (100)(0.200) + (0.16)(60) + (0.128)(20) \\ &= 40\,E(N) - 100 + 20 + 9.6 + 2.56 \\ &= 160 - 67.84 = 92.16\end{aligned}$$

Question #46
Answer: B

$$_{10}E_{30:40} = {}_{10}p_{30}\ {}_{10}p_{40}\ v^{10} = \left({}_{10}p_{30}v^{10}\right)\left({}_{10}p_{40}v^{10}\right)(1+i)^{10}$$
$$= \left({}_{10}E_{30}\right)\left({}_{10}E_{40}\right)(1+i)^{10}$$
$$= (0.54733)(0.53667)(1.79085)$$
$$= 0.52604$$

The above is only one of many possible ways to evaluate ${}_{10}p_{30}\ {}_{10}p_{40}\ v^{10}$, all of which should give 0.52604

$$a_{30:40:\overline{10}|} = a_{30:40} - {}_{10}E_{30:40}\ a_{30+10:40+10}$$
$$= \left(\ddot{a}_{30:40} - 1\right) - (0.52604)\left(\ddot{a}_{40:50} - 1\right)$$
$$= (13.2068) - (0.52604)(11.4784)$$
$$= 7.1687$$

Question #47
Answer: A

Equivalence Principle, where π is annual benefit premium, gives

$$1000\left(A_{35} + (IA)_{35} \times \pi\right) = \ddot{a}_x \pi$$

$$\pi = \frac{1000A_{35}}{\left(\ddot{a}_{35} - (IA)_{35}\right)} = \frac{1000 \times 0.42898}{(11.99143 - 6.16761)}$$
$$= \frac{428.98}{5.82382}$$
$$= 73.66$$

We obtained $\ddot{a}_{35}$ from

$$\ddot{a}_{35} = \frac{1 - A_{35}}{d} = \frac{1 - 0.42898}{0.047619} = 11.99143$$

Question #48
Answer: C

Time until arrival = waiting time plus travel time.

Waiting time is exponentially distributed with mean $1/\lambda$. The time you may already have been waiting is irrelevant: exponential is memoryless.

You: E (wait) = $\frac{1}{20}$ hour = 3 minutes
E (travel) = $(0.25)(16)+(0.75)(28)=25$ minutes
E (total) = 28 minutes

Co-worker: E (wait) = $\frac{1}{5}$ hour = 12 minutes
E (travel) = 16 minutes
E (total) = 28 minutes

Question #49
Answer: C

$$\mu_{xy} = \mu_x + \mu_y = 0.14$$

$$\bar{A}_x = \bar{A}_y = \frac{\mu}{\mu+\delta} = \frac{0.07}{0.07+0.05} = 0.5833$$

$$\bar{A}_{xy} = \frac{\mu_{xy}}{\mu_{xy}+\delta} = \frac{0.14}{0.14+0.05} = \frac{0.14}{0.19} = 0.7368 \text{ and } \bar{a}_{xy} = \frac{1}{\mu_{xy}+\delta} = \frac{1}{0.14+0.05} = 5.2632$$

$$P = \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{xy}} = \frac{\bar{A}_x + \bar{A}_y - \bar{A}_{xy}}{\bar{a}_{xy}} = \frac{2(0.5833)-0.7368}{5.2632} = 0.0817$$

Question #50
Answer: E

$$\left({}_{20}V_{20} + P_{20}\right)(1+i) - q_{40}\left(1 - {}_{21}V_{20}\right) = {}_{21}V_{20}$$

$$(0.49 + 0.01)(1+i) - 0.022(1 - 0.545) = 0.545$$

$$(1+i) = \frac{(0.545)(1-0.022) + 0.022}{0.50}$$

$$= 1.11$$

$$\left({}_{21}V_{20} + P_{20}\right)(1+i) - q_{41}\left(1 - {}_{22}V_{20}\right) = {}_{22}V_{20}$$

$$(0.545+.01)(1.11) - q_{41}(1 - 0.605) = 0.605$$

$$q_{41} = \frac{0.61605 - 0.605}{0.395}$$

$$= 0.028$$

Question #51
Answer: E

$$1000\ P_{60} = 1000\ A_{60} / \ddot{a}_{60}$$

$$= 1000\ v\left(q_{60} + p_{60} A_{61}\right) / \left(1 + p_{60}\ v\ \ddot{a}_{61}\right)$$

$$= 1000\left(q_{60} + p_{60}\ \ A_{61}\right) / \left(1.06 + p_{60}\ \ddot{a}_{61}\right)$$

$$= \left(15 + (0.985)(382.79)\right) / \left(1.06 + (0.985)(10.9041)\right) = 33.22$$

Question #52
Answer: E

Method 1:

In each round,
N = result of first roll, to see how many dice you will roll
X = result of for one of the *N* dice you roll
S = sum of *X* for the *N* dice

$$E(X) = E(N) = 3.5$$
$$Var(X) = Var(N) = 2.9167$$

$$E(S) = E(N) * E(X) = 12.25$$
$$Var(S) = E(N)Var(X) + Var(N)E(X)^2$$
$$= (3.5)(2.9167) + (2.9167)(3.5)^2$$
$$= 45.938$$

Let $S_{1000} =$ the sum of the winnings after 1000 rounds

$$E(S_{1000}) = 1000 * 12.25 = 12{,}250$$
$$Stddev(S_{1000}) = sqrt(1000 * 45.938) = 214.33$$

After 1000 rounds, you have your initial 15,000, less payments of 12,500, plus winnings of S_{1000}.

Since actual possible outcomes are discrete, the solution tests for continuous outcomes greater than 15000-0.5. In this problem, that continuity correction has negligible impact.

$$\Pr(15000 - 12500 + S_{1000} > 14999.5) =$$
$$= \Pr((S_{1000} - 12250)/214.33 > (14999.5 - 2500 - 12250)/214.33) =$$
$$= 1 - \Phi(1.17) = 0.12$$

Method 2

Realize that you are going to determine *N* 1000 times and roll the sum of those 1000 *N*'s dice, adding the numbers showing.

Let $N_{1000} =$ sum of those *N*'s

$$E(N_{1000}) = 1000E(N) = (1000)(3.5) = 3500$$
$$Var(N_{1000}) = 1000Var(N) = 2916.7$$
$$E(S_{1000}) = E(N_{1000})E(X) = (3500)(3.5) = 12.250$$
$$Var(S_{1000}) = E(N_{1000})\,Var(X) + Var(N_{1000})\,E(X)^2$$
$$= (3500)(2.9167) + (2916.7)(3.5)^2 = 45.938$$

$$Stddev(S_{1000}) = 214.33$$

Now that you have the mean and standard deviation of S_{1000} (same values as method 1), use the normal approximation as shown with method 1.

Question #53
Answer: B

$$p_k = \left(a + \frac{b}{k}\right)p_{k-1}$$

$$0.25 = (a+b) \times 0.25 \Rightarrow a + b = 1$$

$$0.1875 = \left(a + \frac{b}{2}\right) \times 0.25 \Rightarrow \left(1 - \frac{b}{2}\right) \times 0.25 = 0.1875$$

$$b = 0.5$$

$$a = 0.5$$

$$p_3 = \left(0.5 + \frac{0.5}{3}\right) \times 0.1875 = 0.125$$

Question #54
Answer: B

Transform these scenarios into a four-state Markov chain, where the final disposition of rates in any scenario is that they decrease, rather than if rates increase, as what is given.

State	from year *t* – 3 to year *t* – 2	from year *t* – 2 to year *t* – 1	Probability that year *t* will decrease from year *t* - 1
0	Decrease	Decrease	0.8
1	Increase	Decrease	0.6
2	Decrease	Increase	0.75
3	Increase	Increase	0.9

Transition matrix is $\begin{bmatrix} 0.80 & 0.00 & 0.20 & 0.00 \\ 0.60 & 0.00 & 0.40 & 0.00 \\ 0.00 & 0.75 & 0.00 & 0.25 \\ 0.00 & 0.90 & 0.00 & 0.10 \end{bmatrix}$

$$P_{00}^2 + P_{01}^2 = 0.8 * 0.8 + 0.2 * 0.75 = 0.79$$

For this problem, you don’t need the full transition matrix. There are two cases to consider. Case 1: decrease in 2003, then decrease in 2004; Case 2: increase in 2003, then decrease in 2004.

For Case 1: decrease in 2003 (following 2 decreases) is 0.8; decrease in 2004 (following 2 decreases is 0.8. Prob(both) = 0.8×0.8 = 0.64
For Case 2: increase in 2003 (following 2 decreases) is 0.2; decrease in 2004 (following a decrease, then increase) is 0.75. Prob(both) = 0.2×0.75 = 0.15
Combined probability of Case 1 and Case 2 is 0.64 + 0.15 = 0.79

Question #55
Answer: B

$$l_x = \omega - x = 105 - x$$
$$\Rightarrow {}_tP_{45} = l_{45+t} / l_{45} = 60 - t / 60$$

Let K be the curtate future lifetime of (45). Then the sum of the payments is 0 if $K \le 19$ and is K – 19 if $K \ge 20$.

$$_{20|}\ddot{a}_{45} = \sum_{K=20}^{60} 1 \times \left(\frac{60-K}{60} \right) \times 1$$

$$= \frac{(40 + 39 + \ldots + 1)}{60} = \frac{(40)(41)}{2(60)} = 13.6\overline{6}$$

Hence,

$$\text{Prob}(K - 19 > 13.6\overline{6}) = \text{Prob}(K > 32.6\overline{6})$$

$$= \text{Prob}(K \ge 33) \text{ since } K \text{ is an integer}$$

$$= \text{Prob}(T \ge 33)$$

$$= {}_{33}p_{45} = \frac{l_{78}}{l_{45}} = \frac{27}{60}$$

$$= 0.450$$

Question #56
Answer: C

$$^2\bar{A}_x = \frac{\mu}{\mu + 2\delta} = 0.25 \rightarrow \mu = 0.04$$

$$\bar{A}_x = \frac{\mu}{\mu + \delta} = 0.4$$

$$\left(I\bar{A}\right)_x = \int_0^\infty {}_{s|}\bar{A}_x \, ds$$

$$\int_0^\infty {}_sE_x \, \bar{A}_x \, ds$$

$$= \int_0^\infty \left(e^{-0.1s}\right)(0.4)\,ds$$

$$= (0.4)\left(\frac{-e^{-0.1s}}{0.1}\right)\Bigg|_0^\infty = \frac{0.4}{0.1} = 4$$

Alternatively, using a more fundamental formula but requiring more difficult integration.

$$\left(I\bar{A}\right)_x = \int_0^\infty t \; {}_tp_x \, \mu_x(t) e^{-\delta t} dt$$

$$= \int_0^\infty t \, e^{-0.04t}(0.04)e^{-0.06t} dt$$

$$= 0.04\int_0^\infty t \, e^{-0.1t} dt$$

(integration by parts, not shown)

$$= 0.04\left(\frac{-t}{0.1} - \frac{1}{0.01}\right)e^{-0.1t}\Big|_0^\infty$$

$$= \frac{0.04}{0.01} = 4$$

Question #57
Answer: E

Subscripts A and B here just distinguish between the tools and do not represent ages.

We have to find $\overset{o}{e}_{\overline{AB}}$

$$\overset{o}{e}_A = \int_0^{10}\left(1-\frac{t}{10}\right)dt = t-\frac{t^2}{20}\Big|_0^{10} = 10-5=5$$

$$\overset{o}{e}_B = \int_0^{7}\left(1-\frac{t}{7}\right)dt = t-\frac{t^2}{14}\Big|_0^{7} = 49-\frac{49}{14}=3.5$$

$$\overset{o}{e}_{AB} = \int_o^{7}\left(1-\frac{t}{7}\right)\left(1-\frac{t}{10}\right)dt = \int_0^{7}\left(1-\frac{t}{10}-\frac{t}{7}+\frac{t^2}{70}\right)dt$$

$$= t-\frac{t^2}{20}-\frac{t^2}{14}+\frac{t^3}{210}\Big|_0^{7}$$

$$= 7-\frac{49}{20}-\frac{49}{14}+\frac{343}{210}=2.683$$

$$\overset{o}{e}_{\overline{AB}} = \overset{o}{e}_A+\overset{o}{e}_B-\overset{o}{e}_{AB}$$

$$=5+3.5-2.683=5.817$$

Question #58
Answer: A

$$\mu_x^{(\tau)}(t) = 0.100 + 0.004 = 0.104$$

$${}_t p_x^{(\tau)} = e^{-0.104t}$$

Actuarial present value (APV) = APV for cause 1 + APV for cause 2.

$$2000\int_0^5 e^{-0.04t} e^{-0.104t}(0.100)dt + 500{,}000\int_0^5 e^{-0.04t}\, e^{-0.104t}(0.400)dt$$

$$= \left(2000(0.10) + 500{,}000(0.004)\right)\int_0^5 e^{-0.144t}dt$$

$$= \frac{2200}{0.144}\left(1 - e^{-0.144(5)}\right) = 7841$$

Question #59
Answer: A

$$R = 1 - p_x = q_x$$

$$S = 1 - p_x \times e^{(-k)} \text{ since } e^{-\int_0^1(\mu_x(t)+k)dt} = e^{-\int_0^1 \mu_x(t)dt - \int_0^1 k\,dt}$$
$$= e^{-\int_0^1 \mu_x(t)dt} e^{-\int_0^1 k\,dt}$$

So $S = 0.75R \Rightarrow 1 - p_x \times e^{-k} = 0.75q_x$

$$e^{-k} = \frac{1 - 0.75q_x}{p_x}$$
$$e^k = \frac{p_x}{1 - 0.75q_x} = \frac{1 - q_x}{1 - 0.75q_x}$$
$$k = \ln\left[\frac{1 - q_x}{1 - 0.75q_x}\right]$$

Question #60
Answer: E

$\beta =$ mean $= 4;\qquad p_k = \beta^k / (1+\beta)^{k+1}$

n	$P(N=n)$
0	0.2
1	0.16
2	0.128
3	0.1024

x	$f^{(1)}(x)$	$f^{(2)}(x)$	$f^{(3)}(x)$
0	0	0	0
1	0.25	0	0
2	0.25	0.0625	0
3	0.25	0.125	0.0156

$f^{(k)}(x) =$ probability that, given exactly *k* claims occur, that the aggregate amount is *x*.

$f^{(1)}(x) = f(x);$ the claim amount distribution for a single claim

$$f^{(k)}(x) = \sum_{j=0}^{x} \left(f^{(k-1)}(j) \right) x\, f(x-j)$$

$f_s(x) = \sum_{k=0}^{x} P(N=k) \times f^{(k)}(x)$; upper limit of sum is really ∞, but here with smallest possible claim size 1, $f^{(k)}(x) = 0$ for $k > x$

$f_s(0) = 0.2$
$f_s(1) = 0.16 * 0.25 = 0.04$
$f_s(2) = 0.16 * 0.25 + 0.128 * 0.0625 = 0.048$
$f_s(3) = 0.16 * 0.25 + 0.128 * 0.125 + 0.1024 * 0.0156 = 0.0576$

$F_s(3) = 0.2 + 0.04 + 0.048 + 0.0576 = 0.346$

Question #61
Answer: E

Let L = incurred losses; P = earned premium = 800,000

Bonus $= 0.15 \times \left(0.60 - \frac{L}{P}\right) \times P$ if positive

$= 0.15 \times (0.60P - L)$ if positive

$= 0.15 \times (480{,}000 - L)$ if positive

$= 0.15 \times (480{,}000 - (L \wedge 480{,}000))$

E (Bonus) = 0.15 (480,000– $E(L \wedge 480{,}000)$)

From Appendix A.2.3.1

= 0.15{480,000 – [500,000 $\times$ (1 – (500,000 / (480,000+500,000)))]}

= 35,265

Question #62
Answer: D

$$\bar{A}^{\,1}_{28:\overline{2}|} = \int_0^2 e^{-\delta t}\, {}^{1}\!/\!_{72}\, dt$$

$$= \frac{1}{72\delta}\left(1 - e^{-2\delta}\right) = 0.02622 \text{ since } \delta = \ln(1.06) = 0.05827$$

$$\ddot{a}_{28:\overline{2}|} = 1 + v\left(\frac{71}{72}\right) = 1.9303$$

$${}_3V = 500{,}000\bar{A}^{\,1}_{28:\overline{2}|} - 6643\,\ddot{a}_{28:\overline{2}|}$$

= 287

Question #63
Answer: D

Let $\bar{A}_x$ and $\bar{a}_x$ be calculated with $\mu_x(t)$ and $\delta = 0.06$

Let $\bar{A}_x^{\,*}$ and $\bar{a}_x^{\,*}$ be the corresponding values with $\mu_x(t)$ increased by 0.03 and δ decreased by 0.03

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{0.4}{0.06} = 6.667$$

$$\bar{a}_x^{\;*} = \bar{a}_x$$

$$\left[\text{Proof: } \bar{a}_x^{\;*} = \int_0^\infty e^{-\int_0^t (\mu_x(s)+0.03)ds} e^{-0.03t} dt\right.$$

$$= \int_0^\infty e^{-\int_0^t \mu_x(s)ds} e^{-0.03t} e^{-0.03t} dt$$

$$= \int_0^\infty e^{-\int_0^t \mu_x(s)ds} e^{-0.06t} dt$$

$$\left.= \bar{a}_x\right]$$

$$\bar{A}_x^{\;*} = 1 - 0.03\bar{a}_x^{\;*} = 1 - 0.03\bar{a}_x$$

$$= 1 - (0.03)(6.667)$$

$$= 0.8$$

Question #64
Answer: A

	bulb ages				
Year	0	1	2	3	# replaced
0	10000	0	0	0	-
1	1000	9000	0	0	1000
2	100+2700	900	6300	0	2800
3	280+270+3150				3700

The diagonals represent bulbs that don't burn out.
E.g., of the initial 10,000, (10,000) (1-0.1) = 9000 reach year 1.
(9000) (1-0.3) = 6300 of those reach year 2.

Replacement bulbs are new, so they start at age 0.
At the end of year 1, that's (10,000) (0.1) = 1000
At the end of 2, it's (9000) (0.3) + (1000) (0.1) = 2700 + 100
At the end of 3, it's (2800) (0.1) + (900) (0.3) + (6300) (0.5) = 3700

Actuarial present value $= \frac{1000}{1.05} + \frac{2800}{1.05^2} + \frac{3700}{1.05^3}$

$= 6688$

Question #65
Key: E

Model Solution:

$$\overset{\circ}{e}_{25:\overline{25}|} = \int_0^{15} {}_t p_{25} dt + {}_{15}p_{25} \int_0^{10} {}_t p_{40} dt$$
$$= \int_0^{15} e^{-.04t} dt + \left(e^{-\int_0^{15} .04 ds} \right) \int_0^{10} e^{-.05t} dt$$
$$= \frac{1}{.04}\left(1 - e^{-.60}\right) + e^{-.60}\left[\frac{1}{.05}\left(1 - e^{-.50}\right) \right]$$
$$= 11.2797 + 4.3187$$
$$= 15.60$$

Question #66
Key: C

Model Solution:

$${}_5 p_{[60]+1} =$$
$$\left(1 - q_{[60]+1}\right)\left(1 - q_{[60]+2}\right)\left(1 - q_{63}\right)\left(1 - q_{64}\right)\left(1 - q_{65}\right)$$
$$= (0.89)(0.87)(0.85)(0.84)(0.83)$$
$$= 0.4589$$

Question # 67
Key: E

Model Solution:

$$12.50 = \bar{a}_x = \frac{1}{\mu + \delta} \Rightarrow \mu + \delta = 0.08 \Rightarrow \mu = \delta = 0.04$$

$$\bar{A}_x = \frac{\mu}{\mu + \delta} = 0.5$$
$${}^2\bar{A}_x = \frac{\mu}{\mu + 2\delta} = \frac{1}{3}$$

$$\text{Var}\left(\bar{a}_{\overline{T}|}\right) = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2}$$

$$= \frac{\frac{1}{3} - \frac{1}{4}}{0.0016} = 52.083$$

S.D. $= \sqrt{52.083} = 7.217$

Question # 68
Key: D

Model Solution:

$$v = 0.90 \Rightarrow d = 0.10$$
$$A_x = 1 - d\ddot{a}_x = 1 - (0.10)(5) = 0.5$$

Benefit premium $\pi = \dfrac{5000A_x - 5000vq_x}{\ddot{a}_x}$

$$= \frac{(5000)(0.5) - 5000(0.90)(0.05)}{5} = 455$$

$${}_{10}V_x = 1 - \frac{\ddot{a}_{x+10}}{\ddot{a}_x}$$

$$0.2 = 1 - \frac{\ddot{a}_{x+10}}{5} \Rightarrow \ddot{a}_{x+10} = 4$$

$$A_{x+10} = 1 - d\ddot{a}_{x+10} = 1 - (0.10)(4) = 0.6$$
$${}_{10}V = 5000A_{x+10} - \pi\ddot{a}_{x+10} = (5000)(0.6) - (455)(4) = 1180$$

Question #68
Key: D

Model Solution:

v is the lowest premium to ensure a zero % chance of loss in year 1 (The present value of the payment upon death is *v*, so you must collect at least *v* to avoid a loss should death occur). Thus *v* = 0.95.

$$\mathrm{E}(\mathrm{Z}) = vq_x + v^2 p_x q_{x+1} = 0.95 \times 0.25 + (0.95)^2 \times 0.75 \times 0.2$$
$$= 0.3729$$
$$\mathrm{E}(\mathrm{Z}^2) = v^2 q_x + v^4 p_x q_{x+1} = (0.95)^2 \times 0.25 + (0.95)^4 \times 0.75 \times 0.2$$
$$= 0.3478$$

$$\mathrm{Var}(\mathrm{Z}) = \mathrm{E}(\mathrm{Z}^2) - (\mathrm{E}(\mathrm{Z}))^2 = 0.3478 - (0.3729)^2 = 0.21$$

Question # 70
Key: D

Model Solution:

Severity after increase	Severity after increase and deductible
60	0
120	20
180	80
300	200

Expected payment per loss $= 0.25 \times 0 + 0.25 \times 20 + 0.25 \times 80 + 0.25 \times 200$
= 75

Expected payments = Expected number of losses $\times$ Expected payment per loss
= 75×300
= 22,500

Question # 71
Key: A

Model Solution:

E (S) = E (N) E (X) = 50 $\times$ 200 = 10,000

$$\mathrm{Var}(\mathrm{S}) = \mathrm{E}(\mathrm{N})\mathrm{Var}(X) + \mathrm{E}(X)^2\,\mathrm{Var}(\mathrm{N})$$
$$= (50)(400) + (200^2)(100)$$
$$= 4{,}020{,}000$$

$$\Pr(S < 8{,}000) = \Pr\left(Z < \frac{8{,}000 - 10{,}000}{\sqrt{4{,}020{,}000}}\right)$$
$$= \Pr(Z < -0.998) \cong 16\%$$

Question #72
Key: A

Model Solution:

Let *Z* be the present value random variable for one life.
Let *S* be the present value random variable for the 100 lives.

$$E(Z) = 10\int_5^\infty e^{\delta t} e^{\mu t} \mu \, dt$$
$$= 10 \frac{\mu}{\delta + \mu} e^{-(\delta+\mu)5}$$
$$= 2.426$$
$$E(Z^2) = 10^2 \left(\frac{\mu}{2\delta + \mu}\right) e^{-(2\delta+\mu)5}$$
$$= 10^2 \left(\frac{0.04}{0.16}\right)\left(e^{-0.8}\right) = 11.233$$
$$Var(Z) = E(Z^2) - (E(Z))^2$$
$$= 11.233 - 2.426^2$$
$$= 5.348$$

$$\mathrm{E(S)} = 100\,\mathrm{E(Z)} = 242.6$$
$$\mathrm{Var(S)} = 100\ \mathrm{Var(Z)} = 534.8$$
$$\frac{F - 242.6}{\sqrt{534.8}} = 1.645 \to F = 281$$

Question #73
Key: D

Model Solution:

Prob{only 1 survives} = 1-Prob{both survive}-Prob{neither survives}

$$= 1 - {}_3p_{50} \times {}_3p_{[50]} - (1 - {}_3p_{50})(1 - {}_3p_{[50]})$$

$$= 1 - \underbrace{(0.9713)(0.9698)(0.9682)}_{=0.912012}\underbrace{(0.9849)(0.9819)(0.9682)}_{0.936320} - (1 - 0.912012)(1 - 0.93632)$$

$$= 0.140461$$

Question # 74
Key: C

Model Solution:

The tyrannosaur dies at the end of the first day if it eats no scientists that day. It dies at the end of the second day if it eats exactly one the first day and none the second day. If it does not die by the end of the second day, it will have at least 10,000 calories then, and will survive beyond 2.5.

Prob (ruin) $= f(0) + f(1)f(0)$

$$= 0.368 + (0.368)(0.368)$$

$$= 0.503$$

since $f(0) = \dfrac{e^{-1}1^0}{0!} = 0.368$

$$f(1) = \frac{e^{-1}1^1}{1!} = 0.368$$

Question #75
Key: B

Model Solution:

Let X = expected scientists eaten.
For each period, $E[X] = E[X|\text{dead}] \times \text{Prob}(\text{already dead}) + E[X|\text{alive}] \times \text{Prob}(\text{alive})$

$$= 0 \times \text{Prob}(\text{dead}) + E[X|\text{alive}] \times \text{Prob}(\text{alive})$$

Day 1, $\text{E}[X_1] = 1$

$\text{Prob}(\text{dead at end of day 1}) = f(0) = \dfrac{e^{-1}0^1}{0!} = 0.368$

Day 2, $\text{E}[X_2] = 0 \times 0.368 + 1 \times (1 - 0.368) = 0.632$

Prob (dead at end of day 2) = 0.503
[per problem 10]

Day 2.5, $\mathrm{E}\left[X_{2.5}\right]=0\times 0.503+0.5\times\left(1-0.503\right)=0.249$
where $E\left[X_{2.5}\middle|\text{alive}\right]=0.5$ since only $\frac{1}{2}$ day in period.

$$\mathrm{E}\left[X\right]=\mathrm{E}\left[X_1\right]+\mathrm{E}\left[X_2\right]+\mathrm{E}\left[X_{2.5}\right]=1+0.632+0.249=1.881$$
$$\mathrm{E}\left[10{,}000X\right]=18{,}810$$

Question # 76
Key: C

Model Solution:

This solution applies the equivalence principle to each life. Applying the equivalence principle to the 100 life group just multiplies both sides of the first equation by 100, producing the same result for *P*.

$$APV\left(\text{Prems}\right)=P=APV\left(\text{Benefits}\right)=10q_{70}v+10\,p_{70}q_{71}v^2+Pp_{70}p_{71}v^2$$
$$P=\frac{(10)(0.03318)}{1.08}+\frac{(10)(1-0.03318)(0.03626)}{1.08^2}+\frac{P(1-0.03318)(1-0.03626)}{1.08^2}$$
$$=0.3072+0.3006+0.7988P$$
$$P=\frac{0.6078}{0.2012}=3.02$$

(*APV* above means Actuarial Present Value).

Question #77
Key: E

Model Solution:

One approach is to recognize an interpretation of formula 7.4.11 or exercise 7.17a:

> Level benefit premiums can be split into two pieces: one piece to provide term insurance
> for *n* years; one to fund the reserve for those who survive.

If you think along those lines, you can derive formula 7.4.11:

$$P_x=P^1_{x:\overline{n}|}+P_{x:\overline{n}|}^{\;\;1}\;{}_nV_x$$

And plug in to get

$$0.090 = P^{1}_{x:\overline{n}|} + (0.00864)(0.563)$$

$$P^{1}_{x:\overline{n}|} = 0.0851$$

Another approach is to think in terms of retrospective reserves. Here is one such solution:

$$
\begin{aligned}
{}_nV_x &= \left(P_x - P^{1}_{x:\overline{n}|}\right)\ddot{s}_{x:\overline{n}|} \\
&= \left(P_x - P^{1}_{x:\overline{n}|}\right)\frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} \\
&= \left(P_x - P^{1}_{x:\overline{n}|}\right)\frac{\ddot{a}_{x:\overline{n}|}}{P_{x:\overline{n}|}^{\ \ 1}\ddot{a}_{x:\overline{n}|}} \\
&= \frac{\left(P_x - P^{1}_{x:\overline{n}|}\right)}{\left(P_{x:\overline{n}|}^{\ \ 1}\right)}
\end{aligned}
$$

$$0.563 = \left(0.090 - P^{1}_{x:\overline{n}|}\right) / 0.00864$$

$$
\begin{aligned}
P^{1}_{x:\overline{n}|} &= 0.090 - (0.00864)(0.563) \\
&= 0.0851
\end{aligned}
$$

Question #78
Key: A

Model Solution:

$$\delta = \ln(1.05) = 0.04879$$

$$
\begin{aligned}
\bar{A}_x &= \int_0^{\omega-x} {}_tp_x \mu_x(t) e^{-\delta t}\,dt \\
&= \int_0^{\omega-x} \frac{1}{\omega-x} e^{-\delta t}\,dt \quad \text{for DeMoivre} \\
&= \frac{1}{\omega-x}\bar{a}_{\overline{\omega-x}|}
\end{aligned}
$$

From here, many formulas for ${}_{10}\bar{V}(\bar{A}_{40})$ could be used. One approach is:

Since

$$\bar{A}_{50} = \frac{\bar{a}_{\overline{50}|}}{50} = \frac{18.71}{50} = 0.3742 \text{ so } \bar{a}_{50} = \left(\frac{1-\bar{A}_{50}}{\delta}\right) = 12.83$$

$$\bar{A}_{40} = \frac{\bar{a}_{\overline{60}|}}{60} = \frac{19.40}{60} = 0.3233 \text{ so } \bar{a}_{40} = \left(\frac{1-\bar{A}_{40}}{\delta}\right) = 13.87$$

$$\text{so } \bar{P}(\bar{A}_{40}) = \frac{0.3233}{13.87} = 0.02331$$

$${}_{10}\bar{V}(\bar{A}_{40}) = \left[\bar{A}_{50} - \bar{P}(\bar{A}_{40})\bar{a}_{50}\right] = \left[0.3742 - (0.02331)(12.83)\right] = 0.0751.$$

Question #79
Key: D

Model Solution:

$$\bar{A}_x = E\left[v^{T(x)}\right] = E\left[v^{T(x)}|NS\right] \times \text{Prob(NS)} + E\left[v^{T(x)}|S\right] \times \text{Prob(S)}$$

$$= \left(\frac{0.03}{0.03+0.08}\right) \times 0.70 + \left(\frac{0.6}{0.06+0.08}\right) \times 0.30$$

$$= 0.3195$$

Similarly, ${}^2\bar{A}_x = \left(\frac{0.03}{0.03+0.16}\right) \times 0.70 + \left(\frac{0.06}{0.06+0.16}\right) \times 0.30 = 0.1923.$

$$\text{Var}\left(\bar{a}_{\overline{T(x)}|}\right) = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2} = \frac{0.1923 - 0.3195^2}{0.08^2} = 14.1.$$

Question #80
Key: B

Model Solution:

Let *S* denote aggregate losses before deductible.

$E[S] = 2 \times 2 = 4$, since mean severity is 2.

$f_S(0) = \dfrac{e^{-2}2^0}{0!} = 0.1353,$ since must have 0 number to get aggregate losses = 0.

$f_S(1) = \left(\dfrac{e^{-2}2}{1!}\right)\left(\dfrac{1}{3}\right) = 0.0902,$ since must have 1 loss whose size is 1 to get aggregate losses = 1.

$$\begin{aligned} E(S \wedge 2) &= 0 \times f_S(0) + 1 \times f_S + 2 \times (1 - f_S(0) - f_S(1)) \\ &= 0 \times 0.1353 + 1 \times 0.0902 + 2 \times (1 - 0.1353 - 0.0902) \\ &= 1.6392 \end{aligned}$$

$$\begin{aligned} E\left[(S-2)_+\right] &= E[S] - E[S \wedge 2] \\ &= 4 - 1.6392 \\ &= 2.3608 \end{aligned}$$

Question #81
Key: D

Model Solution:

Poisson processes are separable. The aggregate claims process is therefore equivalent to two independent processes, one for Type I claims with expected frequency $\left(\dfrac{1}{3}\right)(3000) = 1000$ and one for Type II claims.

Let $S_I =$ aggregate Type I claims.
$N_I =$ number of Type I claims.
$X_I =$ severity of a Type I claim (here = 10).

Since $X_I = 10$, a constant, $E(X_I) = 10$; $\mathrm{Var}(X_I) = 0$.

$$\begin{aligned} \mathrm{Var}(S_I) &= E(N_I)\ \mathrm{Var}(X_I) + \mathrm{Var}(N_I)\left[E(X_I)\right]^2 \\ &= (1000)(0) + (1000)(10)^2 \\ &= 100{,}000 \end{aligned}$$

$$\text{Var}(S) = \text{Var}(S_I) + \text{Var}(S_{II}) \text{ since independent}$$
$$2{,}100{,}000 = 100{,}000 + \text{Var}(S_{II})$$
$$Var(S_{II}) = 2{,}000{,}000$$

Question #82
Key: A

Model Solution:

$$_5 p_{50}^{(\tau)} = {}_5p'^{(1)}_{50}\ {}_5 p'^{(2)}_{50}$$
$$= \left(\frac{100-55}{100-50}\right) e^{-(0.05)(5)}$$
$$= (0.9)(0.7788) = 0.7009$$

Similarly
$$_{10} p_{50}^{(\tau)} = \left(\frac{100-60}{100-50}\right) e^{-(0.05)(10)}$$
$$= (0.8)(0.6065) = 0.4852$$

$$_{5|5}q_{50}^{(\tau)} = {}_5p_{50}^{(\tau)} - {}_{10}p_{50}^{(\tau)} = 0.7009 - 0.4852$$
$$= 0.2157$$

Question #83
Key: C

Model Solution:

Only decrement 1 operates before t = 0.7

$_{0.7}q'^{(1)}_{40} = (0.7)\,q'^{(1)}_{40} = (0.7)(0.10) = 0.07$ since UDD

Probability of reaching t = 0.7 is 1-0.07 = 0.93

Decrement 2 operates only at t = 0.7, eliminating 0.125 of those who reached 0.7

$$q_{40}^{(2)} = (0.93)(0.125) = 0.11625$$

Question #84
Key: C

Model Solution:

$$\pi\left(1+{}_2p_{80}v^2\right)=1000A_{80}+\frac{\pi v q_{80}}{2}+\frac{\pi v^3\,{}_2p_{80}q_{82}}{2}$$

$$\pi\left(1+\frac{0.83910}{1.06^2}\right)=665.75+\pi\left(\frac{0.08030}{2(1.06)}+\frac{0.83910\times 0.09561}{2(1.06)^3}\right)$$

$$\pi(1.74680)=665.75+\pi(0.07156)$$
$$\pi(1.67524)=665.75$$
$$\pi=397.41$$

Where ${}_2p_{80}=\dfrac{3{,}284{,}542}{3{,}914{,}365}=0.83910$

Or ${}_2p_{80}=(1-0.08030)(1-0.08764)=0.83910$

Question #85
Key: E

Model Solution:

At issue, actuarial present value (APV) of benefits

$$=\int_0^\infty b_t v^t\,{}_tp_{65}\,\mu_{65}(t)dt$$
$$=\int_0^\infty 1000\left(e^{0.04t}\right)\left(e^{-0.04t}\right){}_tp_{65}\,\mu_{65}(t)dt$$
$$=1000\int_0^\infty {}_tp_{65}\,\mu_{65}(t)dt=1000\,{}_\infty q_{65}=1000$$

APV of premiums $=\pi\,\bar{a}_{65}=\pi\left(\dfrac{1}{0.04+0.02}\right)=16.667\pi$

Benefit premium $\pi=1000/16.667=60$

$${}_2\bar{V}=\int_0^\infty b_{2+u}\,v^u\,{}_up_{67}\,\mu_{65}(2+u)du-\pi\,\bar{a}_{67}$$
$$=\int_0^\infty 1000\,e^{0.04(2+u)}e^{-0.04u}\,{}_up_{67}\,\mu_{65}(2+u)du-(60)(16.667)$$
$$=1000e^{0.08}\int_0^\infty {}_up_{67}\mu_{65}(2+u)\,du-1000$$
$$=1083.29\,{}_\infty q_{67}-1000=1083.29-1000=83.29$$

Question #86
Key: B

Model Solution:

(1) $a_{x:\overline{20}|} = \ddot{a}_{x:\overline{20}|} - 1 + {}_{20}E_x$

(2) $\ddot{a}_{x:\overline{20}|} = \dfrac{1 - A_{x:\overline{20}|}}{d}$

(3) $A_{x:\overline{20}|} = A^1_{x:\overline{20}|} + A_{x:\overset{1}{\overline{20}|}}$

(4) $A_x = A^1_{x:\overline{20}|} + {}_{20}E_x \ A_{x+20}$

$$0.28 = A^1_{x:\overline{20}|} + (0.25)(0.40)$$

$$A^1_{x:\overline{20}|} = 0.18$$

Now plug into (3): $A_{x:\overline{20}|} = 0.18 + 0.25 = 0.43$

Now plug into (2): $\ddot{a}_{x:\overline{20}|} = \dfrac{1 - 0.43}{(0.05/1.05)} = 11.97$

Now plug into (1): $a_{x:\overline{20}|} = 11.97 - 1 + 0.25 = 11.22$

Question #87
Key: A

Model Solution:

$$E[N] = E_\Lambda\left[E[N|\Lambda]\right] = E_\Lambda[\Lambda] = 2$$

$$\begin{aligned}\text{Var}[N] &= E_\Lambda\left[\text{Var}[N|\Lambda]\right] + \text{Var}_\Lambda\left[E[N|\Lambda]\right] \\ &= E_\Lambda[\Lambda] + Var_\Lambda[\Lambda] = 2 + 4 = 6\end{aligned}$$

Distribution is negative binomial.

Per supplied tables:

$$r\beta = 2 \text{ and } r\beta(1+\beta) = 6$$
$$1+\beta = 3$$
$$\beta = 2$$
$$r\beta = 2 \quad r = 1$$

$$p_1 = \frac{r\beta^1}{1!(1+\beta)^{r+1}} = \frac{(1)(2)}{(1)(3)^2} = 0.22$$

Alternatively, if you don't recognize that *N* will have a negative binomial distribution, derive gamma density from moments (hoping α is an integer).

$$\text{Mean} = \theta\alpha = 2$$

$$Var = E\left[\Lambda^2\right] - E[\Lambda]^2 = \theta^2\left(\alpha^2 + \alpha\right) - \theta^2\alpha^2$$
$$= \theta^2\alpha = 4$$

$$\theta = \frac{\theta^2\alpha}{\theta\alpha} = \frac{4}{2} = 2$$
$$\alpha = \frac{\theta\alpha}{\theta} = 1$$
$$p_1 = \int_0^\infty \left(p_1 | \lambda\right) f(\lambda) d\lambda = \int_0^\infty \frac{e^{-\lambda}\lambda^1}{1!} \frac{(\lambda/2)e^{-(\lambda/2)}}{\lambda\Gamma(1)} d\lambda$$
$$= \frac{1}{2}\int_0^\infty \lambda e^{-\frac{\alpha}{2}} d\lambda$$
[Integrate by parts; not shown]
$$= \frac{1}{2}\left(-\frac{2}{3}\lambda e^{-\frac{3}{2}\lambda} - \frac{4}{9}e^{-\frac{3}{2}\lambda}\right)\Bigg|_0^\infty$$
$$= \frac{2}{9} = 0.22$$

Question #88
Key: C

Model Solution:

Limited expected value =

$$\int_0^{1000} \left(1-F(x)\right)dx = \int_0^{1000} \left(0.8e^{-0.02x} + 0.2e^{-0.001x}\right)dx = \left(-40e^{-0.02x} - 200e^{-0.001x}\right)\Big|_0^{1000} = 40 + 126.4$$

= 166.4

Question #89
Key: E

Model Solution:

$$M = \text{Initial state matrix } = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \text{One year transition matrix } = \begin{bmatrix} 0.20 & 0.80 & 0 & 0 \\ 0.50 & 0 & 0.50 & 0 \\ 0.75 & 0 & 0 & 0.25 \\ 1.00 & 0 & 0 & 0 \end{bmatrix}$$

$$M \times T = \begin{bmatrix} 0.20 & 0.80 & 0 & 0 \end{bmatrix}$$

$$\left(M \times T\right) \times T = \begin{bmatrix} 0.44 & 0.16 & 0.40 & 0 \end{bmatrix}$$

$$\left(\left(M \times T\right) \times T\right) \times T = \begin{bmatrix} 0.468 & 0.352 & 0.08 & 0.10 \end{bmatrix}$$

Probability of being in state *F* after three years = 0.468.

$$\text{Actuarial present value} = \left(0.468v^3\right)\left(500\right) = 171$$

Notes:
1. Only the first entry of the last matrix need be calculated (verifying that the four sum to 1 is useful "quality control.")
2. Compare this with solution 23. It would be valid to calculate T^3 here, but advancing *M* one year at a time seems easier.

Question #90
Key: B

Model Solution:

Let Y_i be the number of claims in the ith envelope.

Let $X(13)$ be the aggregate number of claims received in 13 weeks.

$$E[Y_i] = (1\times 0.2) + (2\times 0.25) + (3\times 0.4) + (4\times 0.15) = 2.5$$
$$E[Y_i^2] = (1\times 0.2) + (4\times 0.25) + (9\times 0.4) + (16\times 0.15) = 7.2$$
$$E[X(13)] = 50\times 13\times 2.5 = 1625$$
$$\text{Var}[X(13)] = 50\times 13\times 7.2 = 4680$$
$$\text{Prob}\{\text{X}(13) \le Z\} = 0.90 = \Phi(1.282)$$
$$\Rightarrow \text{Prob}\left\{\frac{X(13)-1625}{\sqrt{4680}} \le 1.282\right\}$$
$$X(13) \le 1712.7$$

Note: The formula for $\text{Var}[X(13)]$ took advantage of the frequency's being Poisson. The more general formula for the variance of a compound distribution, $\text{Var}(S) = E(N)\ \text{Var}(X) + \text{Var}(N)E(X)^2$, would give the same result.

Question #91
Key: E

Model Solution:

$$\mu^M(60) = \frac{1}{\omega - 60} = \frac{1}{75-60} = \frac{1}{15}$$
$$\mu^F(60) = \frac{1}{\omega' - 60} = \frac{1}{15}\times\frac{3}{5} = \frac{1}{25} \Rightarrow \omega' = 85$$
$${}_t p_{65}^M = 1 - \frac{t}{10}$$
$${}_t p_{60}^F = 1 - \frac{t}{25}$$

Let *x* denote the male and *y* denote the female.

$$\mathring{e}_x = 5 \ (\text{mean for uniform distribution over } (0,10))$$

$$\mathring{e}_y = 12.5 \ (\text{mean for uniform distribution over } (0,25))$$

$$\mathring{e}_{xy} = \int_0^{10}\left(1-\frac{t}{10}\right)\left(1-\frac{t}{25}\right)\cdot dt$$

$$= \int_0^{10}\left(1-\frac{7}{50}t+\frac{t^2}{250}\right)\cdot dt$$

$$= \left(t-\frac{7}{100}t^2+\frac{t^3}{750}\right)\Bigg|_0^{10} = 10-\frac{7}{100}\times 100+\frac{1000}{750}$$

$$= 10-7+\frac{4}{3} = \frac{13}{3}$$

$$\mathring{e}_{\overline{xy}} = \mathring{e}_x+\mathring{e}_y-\mathring{e}_{xy} = 5+\frac{25}{2}-\frac{13}{3} = \frac{30+75-26}{6} = 13.17$$

Question #92
Key: B

Model Solution:

$$\bar{A}_x = \frac{\mu}{\mu+\delta} = \frac{1}{3}$$

$$^2\bar{A}_x = \frac{\mu}{\mu+2\delta} = \frac{1}{5}$$

$$\bar{P}(\bar{A}_x) = \mu = 0.04$$

$$\text{Var}(L) = \left(1+\frac{\bar{P}(\bar{A}_x)}{\delta}\right)^2\left({}^2\bar{A}_x - \bar{A}_x^2\right)$$

$$= \left(1+\frac{0.04}{0.08}\right)^2\left(\frac{1}{5}-\left(\frac{1}{3}\right)^2\right)$$

$$= \left(\frac{3}{2}\right)^2\left(\frac{4}{45}\right)$$

$$= \frac{1}{5}$$

Question #93
Key: B

Model Solution:

$$\text{Mean excess loss} = \frac{E(X) - E(X \wedge 100)}{1 - F(100)}$$
$$= \frac{331 - 91}{0.8} = 300$$

$E(X) = E(X \wedge 1000)$ since $F(1000) = 1.0$

Question #94
Key: E

Model Solution:

Expected insurance benefits per factory $= E\left[(X-1)_+\right]$
$= 0.2 \times 1 + 0.1 \times 2 = 0.4.$

Insurance premium = (1.1) (2 factories) (0.4 per factory) = 0.88.

Let *R* = retained major repair costs, then

$$f_R(0) = 0.4^2 = 0.16$$
$$f_R(1) = 2 \times 0.4 \times 0.6 = 0.48$$
$$f_R(2) = 0.6^2 = 0.36$$

Dividend $= 3 - 0.88 - R - (0.15)(3)$, if positive
$= 1.67 - R$, if positive

$$E(\text{Dividend}) = (0.16)(1.67 - 0) + (0.48)(1.67 - 1) + (0.36)(0) = 0.5888$$

[The (0.36)(0) term in the last line represents that with probability 0.36, $(1.67 - R)$ is negative so the dividend is 0.]

Question #95
Key: A

Model Solution:

$$E[X] = \frac{\alpha\theta}{\alpha - 1} = \frac{4\alpha}{\alpha - 1} = 8 \Rightarrow 4\alpha = 8\alpha - 8$$
$$\alpha = 2$$

$$F(6) = 1 - \left(\frac{\theta}{6}\right)^{\alpha} = 1 - \left(\frac{4}{6}\right)^{2}$$

$$= 0.555$$

$$s(6) = 1 - F(6) = 0.444$$

Question #96
Key: B

Model Solution:

$$e_x = p_x + {}_2p_x + {}_3p_x + \ldots = 11.05$$

$$\text{Annuity} = v^3 \ {}_3p_x 1000 + v^4 \ {}_4p_x \times 1000 \times (1.04) + \ldots$$
$$= \sum_{k=3}^{\infty} 1000(1.04)^{k-3} v^k \ {}_kp_x$$
$$= 1000v^3 \sum_{k=3}^{\infty} {}_k p_x$$
$$= 1000v^3 (e_x - 0.99 - 0.98) = 1000\left(\frac{1}{1.04}\right)^3 \times 9.08 = 8072$$

Let π = benefit premium.

$$\pi\left(1 + 0.99v + 0.98v^2\right) = 8072$$
$$2.8580\pi = 8072$$
$$\pi = 2824$$

Question #97
Key B

Model Solution:

$$\pi \ddot{a}_{30:\overline{10}|} = 1000A_{30} + P(IA)^{1}_{30:\overline{10}|} + (10\pi)\left({}_{10|}A_{30}\right)$$

$$\pi = \frac{1000A_{30}}{\ddot{a}_{30:\overline{10}|} - (IA)^{1}_{30:\overline{10}|} - 10\,{}_{10|}A_{30}}$$

$$= \frac{1000(0.102)}{7.747 - 0.078 - 10(0.088)}$$

$$= \frac{102}{6.789}$$

$$= 15.024$$

Test Question: 98 Key: E

For de Moivre's law,

$$\mathring{e}_{30} = \int_0^{\omega-30}\left(1 - \frac{t}{\omega-30}\right)dt$$

$$= \left[t - \frac{t^2}{2(\omega-30)}\right]_0^{\omega-30}$$

$$= \frac{\omega-30}{2}$$

Prior to medical breakthrough $\omega = 100 \Rightarrow \mathring{e}_{30} = \frac{100-30}{2} = 35$

After medical breakthrough $\mathring{e}'_{30} = \mathring{e}_{30} + 4 = 39$

so $\mathring{e}'_{30} = 39 = \frac{\omega' - 30}{2} \Rightarrow \omega' = 108$

Test Question: 99 Key: A

$$ {}_0L = 100{,}000v^{2.5} - 4000\ddot{a}_{\overline{3}|} \quad @5\% $$

$= 77{,}079$

Test Question: 100 Key: C

$$ \mathrm{E}[\mathrm{N}] = \mathrm{E}_\Lambda\big[\mathrm{E}[\mathrm{N}|\Lambda]\big] = \mathrm{E}_\Lambda[\Lambda] = 2 $$

$$ Var[\mathrm{N}] = \mathrm{E}_\Lambda\big[Var[\mathrm{N}|\Lambda]\big] + Var_\Lambda\big[E[N|\Lambda]\big] $$

$$ = \mathrm{E}_\Lambda[\Lambda] + Var_\Lambda[\Lambda] = 2 + 2 = 4 $$

Distribution is negative binomial (Loss Models, 3.3.2)

Per supplied tables

$$ mean = r\beta = 2 $$

$$ Var = r\beta(1+\beta) = 4 $$

$$ (1+\beta) = 2 $$

$$ \beta = 1 $$

$$ r\beta = 2 $$

$$ r = 2 $$

From tables

$$ p_3 = \frac{r(r+1)(r+2)\beta^3}{3!(1+\beta)^{r+3}} = \frac{(2)(3)(4)1^3}{3!2^5} = \frac{4}{32} = 0.125 $$

1000 $p_3 = 125$

Test Question: 101 **Key: E**

$E[N] = Var[N] = (60)(0.5) = 30$
$E[X] = (0.6)(1) + (0.2)(5) + (0.2)(10) = 3.6$
$E[X^2] = (0.6)(1) + (0.2)(25) + (0.2)(100) = 25.6$
$Var[X] = 25.6 - 3.6^2 = 12.64$

For any compound distribution, per Loss Models
$Var[S] = E[N]Var[X] + Var[N](E[X])^2$

= (30) (12.64) + (30) (3.6^2)

= 768

For specifically Compound Poisson, per Probability Models
$Var[S] = \lambda t\, E[X^2] =$ (60) (0.5) (25.6) = 768

Alternatively, consider this as 3 Compound Poisson processes (coins worth 1; worth 5; worth 10), where for each $Var(X) = 0$, thus for each $Var(S) = Var(N)E[X]^2$.
Processes are independent, so total *Var* is

$$Var = (60)(0.5)(0.6)1^2 + (60)(0.5)(0.2)5^2 + (60)(0.5)(0.2)(10)^2$$
$$= 768$$

Test Question: 102 **Key: D**

$$1000\ {}^{20}_{20}V_x = 1000A_{x+20} = \frac{1000\left({}^{20}_{19}V_x + {}_{20}P_x\right)(1.06) - q_{x+19}(1000)}{p_{x+19}}$$

$$= \frac{(342.03 + 13.72)(1.06) - 0.01254(1000)}{0.98746} = 369.18$$

$$\ddot{a}_{x+20} = \frac{1 - 0.36918}{(0.06/1.06)} = 11.1445$$

$$so \quad 1000P_{x+20} = 1000\frac{A_{x+20}}{\ddot{a}_{x+20}} = \frac{369.18}{11.1445} = 33.1$$

Test Question: 103 Key: B

$$ {}_k p_x^{(\tau)} = e^{-\int_0^k \mu_x^{(\tau)}(t)dt} = e^{-\int_0^k 2\mu_x^{(1)}(t)dt} $$

$$ = \left(e^{-\int_0^k \mu_x^{(1)}(t)dt} \right)^2 $$

$$ = \left({}_k p_x \right)^2 \text{ where } {}_k p_x \text{ is from Illustrative Life Table, since } \mu^{(1)} \text{ follows I.L.T.} $$

$$ {}_{10} p_{60} = \frac{6{,}616{,}155}{8{,}188{,}074} = 0.80802 $$

$$ {}_{11} p_{60} = \frac{6{,}396{,}609}{8{,}188{,}074} = 0.78121 $$

$$ {}_{10|} q_{60}^{(\tau)} = {}_{10} p_{60}^{(\tau)} - {}_{11} p_{60}^{(\tau)} $$

$$ = \left({}_{10} p_{60} \right)^2 - \left({}_{11} p_{60} \right)^2 \text{ from I.L.T.} $$

$$ = 0.80802^2 - 0.78121^2 = 0.0426 $$

Test Question: 104 Key: C

$P_s = \frac{1}{\ddot{a}_s} - d$, where s can stand for any of the statuses under consideration.

$$ \ddot{a}_s = \frac{1}{P_s + d} $$

$$ \ddot{a}_x = \ddot{a}_y = \frac{1}{0.1 + 0.06} = 6.25 $$

$$ \ddot{a}_{\overline{xy}} = \frac{1}{0.06 + 0.06} = 8.333 $$

$$ \ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y $$

$$ \ddot{a}_{xy} = 6.25 + 6.25 - 8.333 = 4.167 $$

$$ P_{xy} = \frac{1}{4.167} - 0.06 = 0.18 $$

Test Question: 105 Key: A

$$d_0^{(\tau)} = 1000\int_0^1 e^{-(\mu+0.04)t}(\mu+0.04)dt$$
$$= 1000\left(1-e^{-(\mu+0.04)}\right) = 48$$

$$e^{-(\mu+0.04)} = 0.952$$
$$\mu + 0.04 = -\ln(0.952)$$
$$= 0.049$$
$$\mu = 0.009$$

$$d_3^{(1)} = 1000\int_3^4 e^{-0.049t}(0.009)dt$$
$$= 1000\frac{0.009}{0.049}\left(e^{-(0.049)(3)} - e^{-(0.049)(4)}\right) = 7.6$$

Test Question: 106 Key: B

This is a graph of $l_x\mu(x)$.
$\mu(x)$ would be increasing in the interval $(80,100)$.
The graphs of $l_x p_x$, l_x and l_x^2 would be decreasing everywhere.
The graph shown is comparable to Figure 3.3.2 on page 65 of Actuarial Mathematics

Test Question: 107 Key: A

Using the conditional mean and variance formulas:

$$E[N] = E_\Lambda(N|\Lambda)$$

$$Var[N] = Var_\Lambda(E(N|\Lambda)) + E_\Lambda(Var(N|\Lambda))$$

Since *N*, given lambda, is just a Poisson distribution, this simplifies to:

$$E[N] = E_\Lambda(\Lambda)$$
$$Var[N] = Var_\Lambda(\Lambda) + E_\Lambda(\Lambda)$$

We are given that $E[N] = 0.2$ and $Var[N] = 0.4$, subtraction gives *Var* (Λ) = 0.2

Test Question: 108 Key: B

N = number of salmon
X = eggs from one salmon
S = total eggs.
$E(N)$ = 100t
$Var(N)$ = 900t

$$E(S)=E(N)E(X)=500t$$

$$Var(S)=E(N)Var(X)+E^2(X)Var(N)=100t\cdot 5+25\cdot 900t=23{,}000t$$

$$P(S>10{,}000)=P\left(\frac{S-500t}{\sqrt{23{,}000t}}>\frac{10{,}000-500t}{\sqrt{23{,}000t}}\right)=.95\Rightarrow$$

$$10{,}000-500t=-1.645\cdot\sqrt{23000}\sqrt{t}=-250\sqrt{t}$$

$$40-2t=-\sqrt{t}$$

$$2\left(\sqrt{t}\right)^2-\sqrt{t}-40=0$$

$$\sqrt{t}=\frac{1\pm\sqrt{1+320}}{4}=4.73$$

$$t=22.4$$

round up to 23

Test Question: 109 Key: A

$$A\ P\ V\ (x\text{'s benefits})=\sum_{k=0}^{2}v^{k+1}b_{k+1}\ {}_kp_x\ q_{x+k}$$

$$=1000\left[300v(0.02)+350v^2(0.98)(0.04)+400v^3(0.98)(0.96)(0.06)\right]$$

$$=36{,}829$$

Test Question: 110 Key: E

π denotes benefit premium

${}_{19}V = APV$ future benefits - APV future premiums

$$0.6 = \frac{1}{1.08} - \pi \Rightarrow \pi = 0.326$$

$$ {}_{11}V = \frac{\left({}_{10}V + \pi\right)(1.08) - \left(q_{65}\right)(10)}{p_{65}}$$

$$= \frac{(5.0 + 0.326)(1.08) - (0.10)(10)}{1 - 0.10}$$

=5.28

Test Question: 111 Key: C

X = losses on one life

$$E[X] = (0.3)(1) + (0.2)(2) + (0.1)(3)$$
$$= 1$$

S = total losses

$$E[S] = 3E[X] = 3$$

$$E\left[(S-1)_+\right] = E[S] - 1\left(1 - F_s(0)\right)$$
$$= E[S] - (1)\left(1 - f_s(0)\right)$$
$$= 3 - (1)\left(1 - 0.4^3\right)$$
$$= 3 - 0.936$$
$$= 2.064$$

Test Question: 112 Key: A

$$1180 = 70\bar{a}_{30} + 50\bar{a}_{40} - 20\bar{a}_{30:40}$$
$$1180 = (70)(12) + (50)(10) - 20\bar{a}_{30:40}$$
$$\bar{a}_{30:40} = 8$$
$$\bar{a}_{\overline{30:40}} = \bar{a}_{30} + \bar{a}_{40} - \bar{a}_{30:40} = 12 + 10 - 8 = 14$$
$$100\bar{a}_{\overline{30:40}} = 1400$$

Test Question: 113 Key: B

$$\bar{a} = \int_0^\infty \bar{a}_{\overline{t}|}\, f(t)dt = \int_0^\infty \frac{1-e^{-0.05t}}{0.05} \frac{1}{\Gamma(2)} te^{-t}dt$$

$$= \frac{1}{0.05}\int_0^\infty \left(te^{-t} - te^{-1.05t}\right)dt$$

$$= \frac{1}{0.05}\left[-(t+1)e^{-t} + \left(\frac{t}{1.05} + \frac{1}{1.05^2}\right)e^{-1.05t}\right]\Bigg|_0^\infty$$

$$= \frac{1}{0.05}\left[1 - \left(\frac{1}{1.05}\right)^2\right] = 1.85941$$

$$20{,}000 \times 1.85941 = 37{,}188$$

Test Question: 114 Key: C

$$p(k) = \frac{2}{k}p(k-1)$$
$$= \left[0 + \frac{2}{k}\right]p(k-1)$$

Thus an (*a*, *b*, 0) distribution with *a* = 0, *b* = 2.

Thus Poisson with $\lambda = 2$.

$$p(4) = \frac{e^{-2}2^4}{4!}$$
$$= 0.09$$

Test Question: 115   Key: B

By the memoryless property, the distribution of amounts paid in excess of 100 is still exponential with mean 200.

With the deductible, the probability that the amount paid is 0 is
$F(100) = 1 - e^{-100/200} = 0.393.$

Thus the average amount paid per loss is (0.393) (0) + (0.607) (200) = 121.4

The expected number of losses is (20) (0.8) = 16.

The expected amount paid is (16) (121.4) = 1942.

Test Question: 116   Key: D

Let M = the force of mortality of an individual drawn at random; and $T =$ future lifetime of the individual.

$$\begin{aligned}
\Pr[T \le 1] &= E\{\Pr[T \le 1 | M]\} \\
&= \int_0^\infty \Pr[T \le 1 | M = \mu]\, f_M(\mu) d\mu \\
&= \int_0^2 \int_0^1 \mu e^{-\mu t} dt \frac{1}{2} d\mu \\
&= \int_0^2 \left(1 - e^{-\mu}\right)\frac{1}{2} du = \frac{1}{2}\left(2 + e^{-2} - 1\right) = \frac{1}{2}\left(1 + e^{-2}\right) \\
&= 0.56767
\end{aligned}$$

Test Question: 117 Key: E

$$E[N] = (0.8)(1) + (0.2)(2) = 1.2$$
$$E[N^2] = (0.8)1 + (0.2)(4) = 1.6$$
$$\text{Var(N)} = 1.6 - 1.2^2 = 0.16$$
$$E[X] = 70 + 100 = 170$$
$$\text{Var(X)} = E[X^2] - E[X]^2 = (7000 + 100{,}000) - 170^2 = 78{,}100$$
$$E[S] = E[N]E[X] = 1.2(170) = 204$$
$$\text{Var(S)} = E[N]\text{Var(X)} + E[X]^2\,\text{Var(N)} = 1.2(78{,}100) + 170^2(0.16) = 98{,}344$$

Std dev $(S) = \sqrt{98{,}344} = 313.6$
So B = 204 + 314 = 518

Test Question: 118 Key: D

Let $\pi =$ benefit premium

Actuarial present value of benefits =

$$= (0.03)(200{,}000)v + (0.97)(0.06)(150{,}000)v^2 + (0.97)(0.94)(0.09)(100{,}000)v^3$$
$$= 5660.38 + 7769.67 + 6890.08$$
$$= 20{,}320.13$$

Actuarial present value of benefit premiums

$$= \ddot{a}_{x:\overline{3}|}\,\pi$$
$$= \left[1 + 0.97v + (0.97)(0.94)v^2\right]\pi$$
$$= 2.7266\,\pi$$
$$\pi = \frac{20{,}320.13}{2.7266} = 7452.55$$

$$_1V = \frac{(7452.55)(1.06) - (200{,}000)(0.03)}{1 - 0.03}$$
$$= 1958.46$$

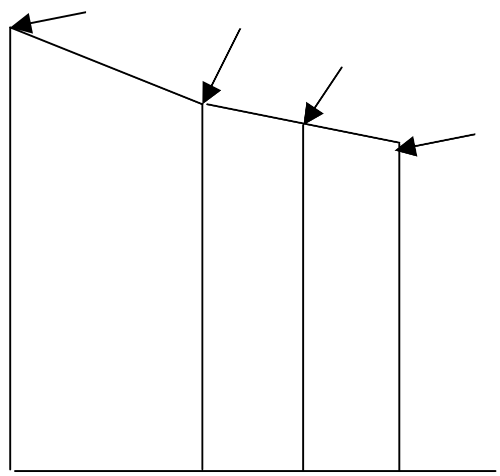
Initial reserve, year 2 = $_1V + \pi$
= 1958.56 + 7452.55
= 9411.01

Test Question: 119 Key: A

Let π denote the premium.

$$L = b_T v^T - \pi \bar{a}_{\overline{T|}} = (1+i)^T \times v^T - \pi \bar{a}_{\overline{T|}}$$
$$= 1 - \pi \bar{a}_{\overline{T|}}$$
$$E[L] = 1 - \pi \bar{a}_x = 0 \quad \Rightarrow \pi = 1/\bar{a}_x$$
$$\Rightarrow L = 1 - \pi \bar{a}_{\overline{T|}} = 1 - \frac{\bar{a}_{\overline{T|}}}{\bar{a}_x} = \frac{\delta \bar{a}_x - (1 - v^T)}{\delta \bar{a}_x}$$
$$= \frac{v^T - (1 - \delta \bar{a}_x)}{\delta \bar{a}_x} = \frac{v^T - \bar{A}_x}{1 - \bar{A}_x}$$

Test Question: 120　　　Key: D



$${}_1p_1 = (1 - 0.1) = 0.9$$
$${}_2p_1 = (0.9)(1 - 0.05) = 0.855$$

since uniform, ${}_{1.5}p_1 = (0.9 + 0.855)/2$
$= 0.8775$

$$\overset{\circ}{e}_{1:\overline{1.5|}} = \text{ Area between } t = 0 \text{ and } t = 1.5$$

$$= \left(\frac{1+0.9}{2}\right)(1) + \left(\frac{0.9+0.8775}{2}\right)(0.5)$$

$$= 0.95 + 0.444$$

$$= 1.394$$

Alternatively,

$$\overset{\circ}{e}_{1:\overline{1.5|}} = \int_0^{1.5} {}_t p_1 dt$$

$$= \int_0^1 {}_t p_1 dt + {}_1 p_1 \int_0^{0.5} {}_x p_2 dx$$

$$= \int_0^1 (1 - 0.1t) dt + 0.9 \int_0^{0.5} (1 - 0.05x) dx$$

$$= \left[t - \tfrac{0.1t^2}{2}\right]\Big|_0^1 + 0.9\left[x - \tfrac{0.05x^2}{2}\right]\Big|_0^{0.5}$$

$$= 0.95 + 0.444 = 1.394$$

Test Question: 121 Key: A

$$10{,}000 A_{63}(1.12) = 5233$$

$$A_{63} = 0.4672$$

$$A_{x+1} = \frac{A_x(1+i) - q_x}{p_x}$$

$$A_{64} = \frac{(0.4672)(1.05) - 0.01788}{1 - 0.01788}$$

$$= 0.4813$$

$$A_{65} = \frac{(0.4813)(1.05) - 0.01952}{1 - 0.01952}$$

$$= 0.4955$$

Single contract premium at 65 = (1.12) (10,000) (0.4955)
= 5550

$$(1+i)^2 = \frac{5550}{5233} \qquad i = \sqrt{\frac{5550}{5233}} - 1 = 0.02984$$

Test Question: 122 **Key: B**

Original Calculation (assuming independence):

$$\mu_x = 0.06$$
$$\mu_y = 0.06$$
$$\mu_{xy} = 0.06 + 0.06 = 0.12$$
$$\bar{A}_x = \frac{\mu_x}{\mu_x + \delta} = \frac{0.06}{0.06 + 0.05} = 0.54545$$
$$\bar{A}_y = \frac{\mu_y}{\mu_y + \delta} = \frac{0.06}{0.06 + 0.05} = 0.54545$$
$$\bar{A}_{xy} = \frac{\mu_{xy}}{\mu_{xy} + \delta} = \frac{0.12}{0.12 + 0.05} = 0.70588$$
$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.54545 + 0.54545 - 0.70588 = 0.38502$$

Revised Calculation (common shock model):

$$\mu_x = 0.06,\ \mu_x^{T^*(x)} = 0.04$$
$$\mu_y = 0.06,\ \mu_y^{T^*(y)} = 0.04$$
$$\mu_{xy} = \mu_x^{T^*(x)} + \mu_y^{T^*(y)} + \mu^Z + 0.04 + 0.04 + 0.02 = 0.10$$
$$\bar{A}_x = \frac{\mu_x}{\mu_x + \delta} = \frac{0.06}{0.06 + 0.05} = 0.54545$$
$$\bar{A}_y = \frac{\mu_y}{\mu_{y+\delta}} = \frac{0.06}{0.06 + 0.05} = 0.54545$$
$$\bar{A}_{xy} = \frac{\mu_{xy}}{\mu_{xy} + \delta} = \frac{0.10}{0.10 + 0.05} = 0.66667$$
$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.54545 + 0.54545 - 0.66667 = 0.42423$$

Difference $= 0.42423 - 0.38502 = 0.03921$

Test Question: 123 **Key: E**

Treat as three independent Poisson variables, corresponding to 1, 2 or 3 claimants.

$$\text{rate}_1 = 6 \quad \left[= \frac{1}{2} \times 12\right]$$
$$\text{rate}_2 = 4$$
$$\text{rate}_3 = 2$$

$$\text{Var}_1 = 6$$
$$\text{Var}_2 = 16 \quad \left[= 4 \times 2^2\right]$$
$$\text{Var}_3 = 18$$

total Var = $6 + 16 + 18 = 40,$ since independent.

Alternatively,

$$E\left(X^2\right) = \frac{1^2}{2} + \frac{2^2}{3} + \frac{3^2}{6} = \frac{10}{3}$$

For compound Poisson, $\text{Var}[S] = E[N]E\left[X^2\right]$

$$= (12)\left(\frac{10}{3}\right) = 40$$

Test Question: 124 **Key: C**

$\int_0^3 \lambda(t)dt = 6$ so $N(3)$ is Poisson with $\lambda = 6.$

P is Poisson with mean 3 (with mean 3 since $\text{Prob}\left(y_i < 500\right) = 0.5$)

P and Q are independent, so the mean of P is 3, no matter what the value of Q is.

Test Question: 125 Key: A

At age *x*:

Actuarial Present value (APV) of future benefits = $\left(\frac{1}{5}A_x\right)1000$

APV of future premiums = $\left(\frac{4}{5}\ddot{a}_x\right)\pi$

$\frac{1000}{5}A_{25} = \frac{4}{5}\pi\,\ddot{a}_{25}$ by equivalence principle

$\frac{1000}{4}\frac{A_{25}}{\ddot{a}_{25}} = \pi \Rightarrow \pi = \frac{1}{4}\times\frac{81.65}{16.2242} = 1.258$

${}_{10}V =$ APV (Future benefits) – APV (Future benefit premiums)

$= \frac{1000}{5}A_{35} - \frac{4}{5}\pi\,\ddot{a}_{35}$

$= \frac{1}{5}(128.72) - \frac{4}{5}(1.258)(15.3926)$

$= 10.25$

Test Question: 126 Key: E

Let Y = present value random variable for payments on one life

$S = \sum Y =$ present value random variable for all payments

$E[Y] = 10\ddot{a}_{40} = 148.166$

$\text{Var}[Y] = 10^2\,\frac{\left({}^2A_{40} - A_{40}^2\right)}{d^2}$

$= 100\left(0.04863 - 0.16132^2\right)(1.06/0.06)^2$

$= 705.55$

$E[S] = 100E[Y] = 14{,}816.6$

$\text{Var}[S] = 100\,\text{Var}[Y] = 70{,}555$

Standard deviation $[S] = \sqrt{70{,}555} = 265.62$

By normal approximation, need

E [*S*] + 1.645 Standard deviations = 14,816.6 + (1.645) (265.62)

= 15,254

Test Question: 127 Key: B

Initial Benefit Prem $= \dfrac{5A_{30} - 4\left(A^{1}_{30:\overline{20|}}\right)}{5\ddot{a}_{30:\overline{35|}} - 4\ddot{a}_{30:\overline{20|}}}$

$$= \frac{5(0.10248) - 4(0.02933)}{5(14.835) - 4(11.959)}$$

$$= \frac{0.5124 - 0.11732}{74.175 - 47.836} = \frac{0.39508}{26.339} = 0.015$$

Where

$$A^{1}_{30:\overline{20|}} = \left(A_{30:\overline{20|}} - A_{30:\overset{1}{\overline{20|}}}\right) = 0.32307 - 0.29374 = 0.02933$$

and

$$\ddot{a}_{30:\overline{20|}} = \frac{1 - A_{30:\overline{20|}}}{d} = \frac{1 - 0.32307}{\left(\frac{0.06}{1.06}\right)} = 11.959$$

Comment: the numerator could equally well have been calculated as $A_{30} + 4\ {}_{20}E_{30}\ A_{50}$
= 0.10248 + (4) (0.29374) (0.24905)
= 0.39510

Test Question: 128 Key: B

$${}_{0.75}p_x = 1 - (0.75)(0.05)$$
$$= 0.9625$$
$${}_{0.75}p_y = 1 - (0.75)(.10)$$
$$= 0.925$$
$${}_{0.75}q_{xy} = 1 - {}_{0.75}p_{xy}$$
$$= 1 - \left({}_{0.75}p_x\right)\left({}_{0.75}p_y\right) \text{ since independent}$$
$$= 1 - (0.9625)(0.925)$$
$$= 0.1097$$

Test Question: 129 Key: A

N = number of physicians E(N) = 3 Var (N) = 2
X = visits per physician E(X) = 30 Var (X) = 30

S = total visits

E(S) = E (N) E(X) = 90

Var(S) = E(N) Var(X) + $E^2(X)$ Var(N) =

= $3 \cdot 30 + 900 \cdot 2 = 1890$

Standard deviation (S) = 43.5

Pr(S>119.5) = Pr$\left(\frac{S-90}{43.5} > \frac{119.5-90}{43.5} \right) = 1 - \Phi(0.68)$ Course 3: November 2000

Test Question: 130 Key: A

The person receives *K* per year guaranteed for 10 years $\Rightarrow K\ddot{a}_{\overline{10|}} = 8.4353K$

The person receives *K* per years alive starting 10 years from now $\Rightarrow {}_{10|}\ddot{a}_{40}K$

*Hence we have $10000 = (8.4353 + {}_{10}E_{40}\ddot{a}_{50})K$

Derive ${}_{10}E_{40}$:

$$A_{40} = A^{1}_{40:\overline{10|}} + ({}_{10}E_{40})A_{50}$$

$${}_{10}E_{40} = \frac{A_{40} - A^{1}_{40:\overline{10|}}}{A_{50}} = \frac{0.30 - 0.09}{0.35} = 0.60$$

Derive $\ddot{a}_{50} = \frac{1 - A_{50}}{d} = \frac{1 - 0.35}{\frac{.04}{1.04}} = 16.90$

Plug in values:

$$10{,}000 = (8.4353 + (0.60)(16.90))K$$

$$= 18.5753K$$

$$K = 538.35$$

Test Question: 131 Key: D

STANDARD: $\mathring{e}_{25:\overline{11|}} = \int_0^{11} \left(1 - \frac{t}{75}\right) dt = t - \frac{t^2}{2 \times 75}\Big|_0^{11} = 10.1933$

MODIFIED: $p_{25} = e^{-\int_0^1 0.1 ds} = e^{-.1} = 0.90484$

$$\mathring{e}_{25:\overline{11}|} = \int_0^1 {}_tp_{25}\,dt + p_{25}\int_0^{10}\left(1-\frac{t}{74}\right)dt$$

$$= \int_0^1 e^{-0.1t}\,dt + e^{-0.1}\int_0^{10}\left(1-\frac{t}{74}\right)dt$$

$$= \frac{1-e^{-0.1}}{0.1} + e^{-0.1}\left(t-\frac{t^2}{2\times 74}\right)\Big|_0^{10}$$

$$= 0.95163 + 0.90484(9.32432) = 9.3886$$

Difference =0.8047

Test Question: 132 Key: B

Comparing B & D: Prospectively at time 2, they have the same future benefits. At issue, B has the lower benefit premium. Thus, by formula 7.2.2, B has the higher reserve.

Comparing A to B: use formula 7.3.5. At issue, B has the higher benefit premium. Until time 2, they have had the same benefits, so B has the higher reserve.

Comparing B to C: Visualize a graph C* that matches graph B on one side of t=2 and matches graph C on the other side. By using the logic of the two preceding paragraphs, C's reserve is lower than C*'s which is lower than B's.

Comparing B to E: Reserves on E are constant at 0.

Test Question: 133 Key: C

Since only decrements (1) and (2) occur during the year, probability of reaching the end of the year is

$$p'^{(1)}_{60} \times p'^{(2)}_{60} = (1-0.01)(1-0.05) = 0.9405$$

Probability of remaining through the year is

$$p'^{(1)}_{60} \times p'^{(2)}_{60} \times p'^{(3)}_{60} = (1-0.01)(1-0.05)(1-0.10) = 0.84645$$

Probability of exiting at the end of the year is

$$q^{(3)}_{60} = 0.9405 - 0.84645 = 0.09405$$

Test Question: 134 Key: E

$\mathrm{E}(N) = .7$

$\mathrm{Var}(N) = 4\times.2 + 9\times.1 - .49 = 1.21$

$\mathrm{E}(X) = 2$

$\mathrm{Var}(X) = 100\times.2 - 4 = 16$

$\mathrm{E}(S) = 2\times.7 = 1.4$

$\mathrm{Var}(S) = \mathrm{E}(N)\ \mathrm{Var}(X) + \mathrm{E}^2(X)\ \mathrm{Var}(N) =$

$= .7\times 16 + 4\times 1.21 = 16.04$

$\text{Standard Dev}(S) = 4$

$\mathrm{E}(S) + 2\times\text{Standard Dev}(S) = 1.4 + 2\times 4 = 9.4$

Since there are no possible values of *S* between 0 and 10,

$\Pr(S > 9.4) = 1 - \Pr(S = 0)$

$= 1 - .7 - .2\times.8^2 - .1\times.8^3 = .12$

Test Question: 135 Key: D

APV of regular death benefit $= \int_0^\infty (100000)\left(e^{-\delta t}\right)(0.008)\left(e^{-\mu t}\right)dt$

$= \int_0^\infty (100000)\left(e^{-0.06t}\right)(0.008)\left(e^{-0.008t}\right)dt$

$= 100000\left[0.008 / (0.06 + 0.008)\right] = 11{,}764.71$

APV of accidental death benefit $= \int_0^{30} (100000)\left(e^{-\delta t}\right)(0.001)\left(e^{-\mu t}\right)dt$

$= \int_0^{30} (100000)\left(e^{-0.06t}\right)(0.001)\left(e^{-0.008t}\right)dt$

$= 100\left[1 - e^{-2.04}\right] / 0.068 = 1{,}279.37$

Total APV $= 11765 + 1279 = 13044$

Test Question: 136 Key: B

$$l_{[60]+.6} = (.6)(79{,}954) + (.4)(80{,}625)$$
$$= 80{,}222.4$$
$$l_{[60]+1.5} = (.5)(79{,}954) + (.5)(78{,}839)$$
$$= 79{,}396.5$$

$${}_{0.9}q_{[60]+.6} = \frac{80222.4 - 79{,}396.5}{80{,}222.4}$$
$$= 0.0103$$

$$P_0 = \tfrac{1}{11} = 9.09\overline{09}\%$$

Test Question: 137 Key: A

$$P(0) = \tfrac{1}{5}\int_0^5 e^{-\lambda} d\lambda = \tfrac{1}{5}\left(-e^{-\lambda}\right)\Big|_0^5 = \tfrac{1}{5}\left(1 - e^{-5}\right) = 0.1987$$

$$P(1) = \tfrac{1}{5}\int_0^5 \lambda e^{-\lambda} d\lambda = \tfrac{1}{5}\left(-\lambda e^{-\lambda} - e^{-\lambda}\right)\Big|_0^5 = \tfrac{1}{5}\left(1 - 6e^{-5}\right) = 0.1919$$

$$P(N \ge 2) = 1 - .1987 - .1919 = .6094$$

Test Question: 138 Key: A

$$q_{40}^{(\tau)} = q_{40}^{(1)} + q_{40}^{(2)} = 0.34$$
$$= 1 - p_{40}'^{(1)} p_{40}'^{(2)}$$
$$0.34 = 1 - 0.75 p_{40}'^{(2)}$$

$$p_{40}'^{(2)} = 0.88$$
$$q_{40}'^{(2)} = 0.12 = y$$

$$q_{41}'^{(2)} = 2y = 0.24$$

$q_{41}^{(\tau)} = 1 - (0.8)(1 - 0.24) = 0.392$
$l_{42}^{(\tau)} = 2000(1 - 0.34)(1 - 0.392) = 803$

Test Question: 139 Key: C

$\Pr[L(\pi') > 0] < 0.5$
$\Pr\left[10{,}000 v^{K+1} - \pi' \ddot{a}_{\overline{K+1|}} > 0\right] < 0.5$
From Illustrative Life Table, ${}_{47}p_{30} = 0.50816$ and ${}_{48}p_{30} = .47681$

Since *L* is a decreasing function of *K*, to have
$\Pr[L(\pi') > 0] < 0.5$ means we must have $L(\pi') \le 0$ for $K \ge 47.$
Highest value of $L(\pi')$ for $K \ge 47$ is at *K* = 47.
$L(\pi')[\text{at } K = 47] = 10{,}000\, v^{47+1} - \pi'\, \ddot{a}_{\overline{47+1|}}$
$= 609.98 - 16.589\pi'$
$L(\pi') \le 0 \Rightarrow (609.98 - 16.589\pi') \le 0$
$\Rightarrow \pi' > \dfrac{609.98}{16.589} = 36.77$

Test Question: 140 Key: B

$\Pr(K = 0) = 1 - p_x = 0.1$
$\Pr(K = 1) = {}_1p_x - {}_2p_x = 0.9 - 0.81 = 0.09$
$\Pr(K > 1) = {}_2p_x = 0.81$

$E(Y) = .1 \times 1 + .09 \times 1.87 + .81 \times 2.72 = 2.4715$
$E(Y^2) = .1 \times 1^2 + .09 \times 1.87^2 + .81 \times 2.72^2 = 6.407$
$VAR(Y) = 6.407 - 2.4715^2 = 0.299$

Test Question: 141 Key: D

Let X be the occurrence amount, Y = max(X-100, 0) be the amount paid.
E[X] = 1,000
Var[X] = $(1{,}000)^2$
P(X>100) = exp(-100/1,000) = .904837

The distribution of Y given that X>100, is also exponential with mean 1,000 (memoryless property).

So Y is $\begin{cases} 0 \text{ with prob } .095163 \\ \text{exponential mean 1000 with prob } .904837 \end{cases}$

$$\mathrm{E}[Y] = .095163 \times 0 + .904837 \times 1{,}000 = 904.837$$
$$\mathrm{E}[Y^2] = .095163 \times 0 + .904837 \times 2 \times (1{,}000)^2 = 1{,}809{,}675$$

$$\mathrm{Var}[Y] = 1{,}809{,}675 - (904.837)^2 = 990{,}944$$

Alternatively, think of this as a compound distribution whose frequency is Bernoulli with p = .904837, and severity is exponential with mean 1,000.

$$\mathrm{Var} = \mathrm{Var}[N] \times \mathrm{E}[X]^2 + \mathrm{Var}[X] \times \mathrm{E}[N] = p(1-p)(1{,}000{,}000) + p(1{,}000{,}000)$$

Test Question: 142 Key: B

In general $\mathrm{Var}(L) = \left(1+\frac{P}{\delta}\right)^2\left({}^2\overline{A}_x - \overline{A}_x^2\right)$

Here $\overline{P}(\overline{A}_x) = \dfrac{1}{\overline{a}_x} - \delta = \dfrac{1}{5} - .08 = .12$

So $\mathrm{Var}(L) = \left(1+\dfrac{.12}{.08}\right)^2\left({}^2\overline{A}_x - \overline{A}_x^2\right) = .5625$

and $\mathrm{Var}(L^*) = \left(1+\dfrac{\frac{5}{4}(.12)}{.08}\right)^2\left({}^2\overline{A}_x - \overline{A}_x^2\right)$

So $\mathrm{Var}(L^*) = \dfrac{\left(1+\frac{15}{8}\right)^2}{\left(1+\frac{12}{8}\right)^2}(0.5625) = .744$

$\mathrm{E}[L^*] = \overline{A}_x - .15\overline{a}_x = 1 - \overline{a}_x(\delta + .15) = 1 - 5(.23) = -.15$

$\mathrm{E}[L^*] + \sqrt{\mathrm{Var}(L^*)} = .7125$

Test Question: 143 Key: C

Serious claims are reported according to a Poisson process at an average rate of 2 per month. The chance of seeing at least 3 claims is (1 – the chance of seeing 0, 1, or 2 claims).

$P(3+) \geq 0.9$ is the same as $P(0,1,2) \leq 0.1$ is the same as $\left[P(0)+P(1)+P(2)\right] \leq 0.1$

$0.1 \geq e^{-\lambda} + \lambda e^{-\lambda} + \left(\lambda^2/2\right)e^{-\lambda}$

The expected value is 2 per month, so we would expect it to be at least 2 months $(\lambda = 4)$.

Plug in and try

$e^{-4} + 4e^{-4} + \left(4^2/2\right)e^{-4} = .238,$ too high, so try 3 months $(\lambda = 6)$

$e^{-6} + 6e^{-6} + \left(6^2/2\right)e^{-6} = .062,$ okay. The answer is 3 months.

[While 2 is a reasonable first guess, it was not critical to the solution. Wherever you start, you should conclude 2 is too few, and 3 is enough].

Test Question: 144 Key: B

Let $l_0^{(\tau)} =$ number of students entering year 1

superscript (f) denote academic failure

superscript (w) denote withdrawal

subscript is "age" at start of year; equals year - 1

$$p_0^{(\tau)} = 1 - 0.40 - 0.20 = 0.40$$

$$l_2^{(\tau)} = 10\, l_2^{(\tau)}\, q_2^{(f)} \Rightarrow q_2^{(f)} = 0.1$$

$$q_2^{(w)} = q_2^{(\tau)} - q_2^{(f)} = (1.0 - 0.6) - 0.1 = 0.3$$

$$l_1^{(\tau)} q_1^{(f)} = 0.4\left[l_1^{(\tau)}\left(1 - q_1^{(f)} - q_1^{(w)}\right)\right]$$

$$q_1^{(f)} = 0.4\left(1 - q_1^{(f)} - 0.3\right)$$

$$q_1^{(f)} = \frac{0.28}{1.4} = 0.2$$

$$p_1^{(\tau)} = 1 - q_1^{(f)} - q_1^{(w)} = 1 - 0.2 - 0.3 = 0.5$$

$${}_3q_0^{(w)} = q_0^{(w)} + p_0^{(\tau)} q_1^{(w)} + p_0^{(\tau)} p_1^{(\tau)} q_2^{(w)}$$

$$= 0.2 + (0.4)(0.3) + (0.4)(0.5)(0.3)$$

$$= 0.38$$

Test Question: 145 Key: D

$$e_{25} = p_{25}(1+e_{26})$$

$$e_{26}^N = e_{26}^M \quad \text{since same } \mu$$

$$p_{25}^N = e^{-\int_0^1 \left[\mu_{25}^M(t)+0.1(1-t)\right]dt}$$
$$= e^{-\int_0^1 \mu_{25}^M(t)dt - \int_0^1 0.1(1-t)dt}$$

$$= e^{-\int_0^1 \mu_{25}^M(t)dt} e^{-\int_0^1 0.1(1-t)dt}$$

$$= p_{25}^M e^{-\left[0.1\left(t-\frac{t^2}{2}\right)\right]_0^1}$$

$$= e^{-0.05} p_{25}^M$$

$$e_{25}^N = p_{25}^N(1+e_{26})$$

$$= e^{-0.05} p_{25}^M (1+e_{26})$$

$$= 0.951\, e_{25}^M = (0.951)(10.0) = 9.5$$

Test Question: 146 Key: D

$$\mathrm{E}[Y_{AGG}] = 100\mathrm{E}[Y] = 100(10{,}000)\bar{a}_x$$
$$= 100(10{,}000)\left(\frac{(1-\bar{A}_x)}{\delta}\right) = 10{,}000{,}000$$

$$\sigma_Y = \sqrt{\mathrm{Var}[Y]} = \sqrt{(10{,}000)^2 \frac{1}{\delta^2}\left({}^2\bar{A}_x - \bar{A}_x^2\right)}$$
$$= \frac{(10{,}000)}{\delta}\sqrt{(0.25)-(0.16)} = 50{,}000$$

$$\sigma_{AGG} = \sqrt{100}\sigma_Y = 10(50{,}000) = 500{,}000$$

$$0.90 = \Pr\left[\frac{F - \mathrm{E}[Y_{AGG}]}{\sigma_{AGG}} > 0\right]$$
$$\Rightarrow 1.282 = \frac{F - \mathrm{E}[Y_{AGG}]}{\sigma_{AGG}}$$
$$F = 1.282\sigma_{AGG} + \mathrm{E}[Y_{AGG}]$$
$$F = 1.282(500{,}000) + 10{,}000{,}000 = 10{,}641{,}000$$

Test Question: 147 Key: C

Expected claims under current distribution = 500
$\theta =$ parameter of new distribution
$X =$ claims
$\mathrm{E}(X) = \theta$
bonus $= .5 \times [500 - X \wedge 500]$
E (claims + bonus) $= \theta + .5\left(500 - \theta\left(1 - \frac{\theta}{500+\theta}\right)\right) = 500$

$$\theta - \frac{\theta}{2}\left(\frac{500}{500+\theta}\right) = 250$$
$$2(500+\theta)\theta - 500\theta = 250(500+\theta)\cdot 2$$
$$1000\theta + \theta^2 \cdot 2 - 500\theta = 2 \times 250 \times 500 + 500\theta$$
$$\theta = \sqrt{250 \times 500} = 354$$

Test Question: 148 Key: E

$$(DA)^1_{80:\overline{20}|} = 20vq_{80} + vp_{80}\left((DA)^1_{81:\overline{19}|}\right)$$

$q_{80} = .2$

$$13 = \frac{20(.2)}{1.06} + \frac{.8}{1.06}(DA)^1_{81:\overline{19}|}$$

$$\therefore (DA)^1_{81:\overline{19}|} = \frac{13(1.06)-4}{.8} = 12.225$$

$q_{80} = .1$

$$DA^1_{80:\overline{20}|} = 20v(.1) + v(.9)(12.225)$$

$$= \frac{2+.9(12.225)}{1.06} = 12.267$$

Test Question: 149 Key: B

Let *T* denote the random variable of time until the college graduate finds a job
Let $\{N(t), t \geq 0\}$ denote the job offer process

Each offer can be classified as either

$$\begin{cases} \text{Type I - - accept with probability } p \Rightarrow \{N_1(t)\} \\ \text{Type II - - reject with probability } (1-p) \Rightarrow \{N_2(t)\} \end{cases}$$

By proposition 5.2, $\{N_1(t)\}$ is Poisson process with $\lambda_1 = \lambda \cdot p$

$$p = \Pr(w > 28{,}000) = \Pr(\ln w > \ln 28{,}000)$$

$$= \Pr(\ln w > 10.24) = \Pr\left(\frac{\ln w - 10.12}{0.12} > \frac{10.24 - 10.12}{0.12}\right) = 1 - \Phi(1)$$

$$= 0.1587$$

$$\lambda_1 = 0.1587 \times 2 = 0.3174$$

T has an exponential distribution with $\theta = \frac{1}{.3174} = 3.15$

$$\Pr(T > 3) = 1 - F(3)$$

$$= e^{\frac{-3}{3.15}} = 0.386$$

Test Question: 150 Key: A

$${}_t p_x = \exp\left[-\int_0^t \frac{ds}{100-x-s}\right] = \exp\left[\ln(100-x-s)\Big|_0^t\right] = \frac{100-x-t}{100-x}$$

$$\overset{\circ}{e}_{\overline{50:60}} = \overset{\circ}{e}_{50} + \overset{\circ}{e}_{60} - \overset{\circ}{e}_{50:60}$$

$$\overset{\circ}{e}_{50} = \int_0^{50} \frac{50-t}{50}\,dt = \frac{1}{50}\left[50t - \frac{t^2}{2}\right]_0^{50} = 25$$

$$\overset{\circ}{e}_{60} = \int_0^{40} \frac{40-t}{40}\,dt = \frac{1}{40}\left[40t - \frac{t^2}{2}\right]_0^{40} = 20$$

$$\overset{\circ}{e}_{50:60} = \int_0^{40} \left(\frac{50-t}{50}\right)\left(\frac{40-t}{40}\right)dt = \int_0^{40} \frac{1}{2000}\left(2000 - 90t + t^2\right)dt$$

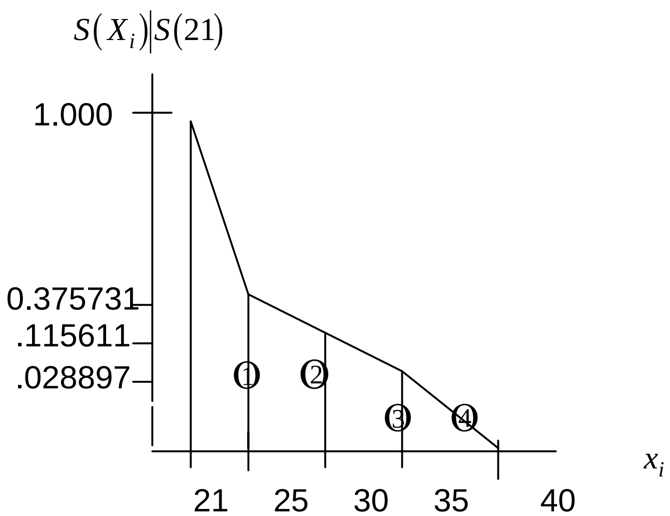
$$= \frac{1}{2000}\left(2000t - 45t^2 + \frac{t^3}{3}\Big|_0^{40}\right) = 14.67$$

$$\overset{\circ}{e}_{\overline{50:60}} = 25 + 20 - 14.67 = 30.33$$

Test Question: 151 Key: A

$UDD \Rightarrow l_{21} = (0.8)(53{,}488) + (0.2)(17{,}384) = 46{,}267.2$

$$Mrl(21) = \overset{\circ}{e}_{21} = \int_0^\infty {}_t p_{21} \int_0^\infty \frac{S(21+t)}{S(21)} dt$$
$$= \sum \text{areas}$$
$$= 2.751 + 1.228 + 0.361 + 0.072$$
$$= 4.412$$



Test: 152 Key: Question D

$\mu'_{n1} = \mathrm{E}[N] = 25$
$\mu_{x2} = \mathrm{Var}[X] = 675$
$\mu_{N2} = \mathrm{Var}[N] = 25$
$E[X] = 50$

$\mathrm{E}[S] = \mathrm{E}[X]\mathrm{E}[N] = 25 \times 50 = 1250$

$$\mathrm{Var}[S] = E[N]\mathrm{Var}[X] + \mathrm{Var}[N]E[X]^2$$
$$= 25 \times 675 + 25 \times 2500 = 79{,}375$$

Standard Deviation$[S] = \sqrt{79,375} = 281.74$

$$\Pr(S > 2000) = \Pr[(S - 1250)/281.74 > (2000 - 1250)/281.74] = 1 - \Phi(2.66)$$

Test Question: 153 Key: E

$$\text{Var}({}_0L) = \text{Var}(\Lambda_0) + v^2\,\text{Var}(\Lambda_1) \quad \text{since } Var(\Lambda_2) = 0$$

$$\begin{aligned}\text{Var}(\Lambda_0) &= [v(b_1 - {}_1V)]^2\, p_{50} q_{50} \\ &= \frac{(10,000 - 3,209)^2 (0.00832)(0.99168)}{1.03^2} \\ &= 358664.09\end{aligned}$$

$$\begin{aligned}\text{Var}(\Lambda_1) &= [v(b_2 - {}_2V)]^2\, p_{50} q_{51} p_{51} \\ &= \frac{(10,000 - 6,539)^2 (0.99168)(0.00911)(0.99089)}{1.03^2} \\ &= 101075.09\end{aligned}$$

$$\text{Var}({}_0L) = 358664.09 + \frac{101075.09}{1.03^2} = 453937.06$$

Alternative solution:

$$\pi = 10,000\,v - {}_2V_{50:\overline{3|}} = 9708.74 - 6539 = 3169.74$$

$$ {}_0L = \begin{cases} 10,000\,v - \pi\,\ddot{a}_{\overline{1|}} = 6539 & \text{for} \quad K = 0 \\ 10,000\,v^2 - \pi\,\ddot{a}_{\overline{2|}} = 3178.80 & \text{for} \quad K = 1 \\ 10,000\,v^3 - \pi\,\ddot{a}_{\overline{3|}} = -83.52 & \text{for} \quad K > 1 \end{cases}$$

$\Pr(K=0) = q_{50} = 0.00832$

$\Pr(K=1) = p_{50}\, q_{51} = (0.99168)(0.00911) = 0.0090342$

$\Pr(K>1) = 1 - \Pr(K=0) - \Pr(K=1) = 0.98265$

$\mathrm{Var}({}_0L) = \mathrm{E}\left[{}_0L^2\right] - \mathrm{E}\left[{}_0L\right]^2 = \mathrm{E}\left[{}_0L^2\right]$ since π is benefit premium

$= 0.00832 \times 6539^2 + 0.00903 \times 3178.80^2 + 0.98265 \times (-83.52)^2$

$= 453{,}895$ [difference from the other solution is due to rounding]

Test Question: 154 Key: C

Let π denote the single benefit premium.

$$\pi = {}_{30|}\ddot{a}_{35} + \pi A^{1}_{35:\overline{30|}}$$

$$\pi = \frac{{}_{30|}\ddot{a}_{35}}{1 - A^{1}_{35:\overline{30|}}} = \frac{\left(A_{35:\overline{30|}} - A^{1}_{35:\overline{30|}}\right)\ddot{a}_{65}}{1 - A^{1}_{35:\overline{30|}}} =$$

$$= \frac{(.21 - .07)9.9}{(1 - .07)}$$

$$= \frac{1.386}{.93}$$

$$= 1.49$$

Test Question: 155 Key: E

$${}_{0.4}p_0 = .5 = e^{-\int_0^{0.4}\left(F + e^{2x}\right)dx}$$

$$= e^{-.4F - \left[\frac{e^{2x}}{2}\right]_0^{.4}}$$

$$= e^{-.4F - \left(\frac{e^{0.8} - 1}{2}\right)}$$

$$.5 = e^{-.4F - .6128}$$

$\Rightarrow \ln(.5) = -.4F - .6128$

$\Rightarrow -.6931 = -.4F - .6128$

$\Rightarrow F = 0.20$

Test Question: 156 Key: C

$$\mathrm{E}\left[X\right] = 2000(1!)/(1!) = 2000$$

$$\mathrm{E}\left[X \wedge 3000\right] = \left(\frac{2000}{1}\right) \times \left[1 - \frac{2000}{(3000+2000)}\right] = 2000 \times \left(1 - \frac{2}{5}\right) = 2000 \times \frac{3}{5} = 1200$$

So the fraction of the losses expected to be covered by the reinsurance is $\dfrac{2000-1200}{2000} = 0.4.$

The expected ceded losses are 4,000,000 $\Rightarrow$ the ceded premium is 4,400,000.

Test Question: 157 Key: E

$$X_{2002} = 1.05 \times X_{2001}$$

so: $$F\left(\frac{x_{2002}}{1.05}\right) = 1 - \left[\frac{2000}{(x_{2002}/1.05 + 2000)}\right]^2$$

$$= 1 - \left[\frac{2100}{x_{2002} + 2100}\right]^2$$

This is just another Pareto distribution with $\alpha = 2, \theta = 2100.$

$\mathrm{E}\left[X_{2002}\right] = 2100.$

and

$$\mathrm{E}\left[X_{2002} \wedge 3000\right] = \left(\frac{2100}{1}\right) \times \left[1 - \left(\frac{2100}{(3000+2100)}\right)\right]$$

$$= 2100 \times \left[\frac{3000}{5100}\right] = 1235$$

So the fraction of the losses expected to be covered by the reinsurance is

$$\frac{2100-1235}{2100} = 0.412.$$

The total expected losses have increased to 10,500,000, so

$$C_{2002} = 1.1 \times 0.412 \times 10{,}500{,}000 = 4{,}758{,}600$$

And $\frac{C_{2002}}{C_{2001}} = \frac{4{,}758{,}600}{4{,}400{,}000} = 1.08$