Course 1 Solutions
November 2001 Exams

1. A

For $i = 1, 2$, let

R_i = event that a red ball is drawn form urn i

B_i = event that a blue ball is drawn from urn i .

Then if x is the number of blue balls in urn 2,

$$
\begin{aligned}
0.44 &= \Pr[(R_1 \cap R_2) \cup (B_1 \cap B_2)] = \Pr[R_1 \cap R_2] + \Pr[B_1 \cap B_2] \\
&= \Pr[R_1]\Pr[R_2] + \Pr[B_1]\Pr[B_2] \\
&= \frac{4}{10}\left(\frac{16}{x+16}\right) + \frac{6}{10}\left(\frac{x}{x+16}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2.2 &= \frac{32}{x+16} + \frac{3x}{x+16} = \frac{3x+32}{x+16} \\
2.2x + 35.2 &= 3x + 32 \\
0.8x &= 3.2 \\
x &= 4
\end{aligned}
$$

2. C

We are given that

$$\int_R\int f(x, y)\, dA = 6 \quad \text{and} \quad \int_R\int dA = 2$$

It follows that

$$
\begin{aligned}
\int_R\int \left[4f(x, y) - 2\right] dA &= 4\int_R\int f(x, y)\, dA - 2\int_R\int dA \\
&= 4(6) - 2(2) = 20
\end{aligned}
$$

3. A

Since $S = 175 L^{3/2} A^{4/5}$, we have

$$\frac{\partial S}{\partial L} = 262.5\ L^{1/2} A^{4/5} > 0 \quad \text{for} \quad L > 0\ ,\ A > 0$$

$$\frac{\partial^2 S}{\partial L^2} = 131.25\ L^{-1/2} A^{4/5} > 0 \quad \text{for} \quad L > 0\ ,\ A > 0$$

$$\frac{\partial S}{\partial A} = 140\ L^{3/2} A^{-1/5} > 0 \quad \text{for} \quad L > 0\ ,\ A > 0$$

$$\frac{\partial^2 S}{\partial A^2} = -28\ L^{3/2} A^{-6/5} < 0 \quad \text{for} \quad L > 0\ ,\ A > 0$$

It follows that S increases at an increasing rate as L increases, while S increases at a decreasing rate as A increases.

4. B

Apply Baye's Formula:

$$\Pr\left[\text{Seri.}\middle|\text{Surv.}\right]$$

$$= \frac{\Pr\left[\text{Surv.}\middle|\text{Seri.}\right]\Pr\left[\text{Seri.}\right]}{\Pr\left[\text{Surv.}\middle|\text{Crit.}\right]\Pr\left[\text{Crit.}\right] + \Pr\left[\text{Surv.}\middle|\text{Seri.}\right]\Pr\left[\text{Seri.}\right] + \Pr\left[\text{Surv.}\middle|\text{Stab.}\right]\Pr\left[\text{Stab.}\right]}$$

$$= \frac{(0.9)(0.3)}{(0.6)(0.1) + (0.9)(0.3) + (0.99)(0.6)} = 0.29$$

5. A

Let us first determine K. Observe that

$$1 = K\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) = K\left(\frac{60+30+20+15+12}{60}\right) = K\left(\frac{137}{60}\right)$$

$$K = \frac{60}{137}$$

It then follows that

$$\Pr[N=n] = \Pr[N=n \mid \text{Insured Suffers a Loss}]\Pr[\text{Insured Suffers a Loss}]$$

$$= \frac{60}{137N}(0.05) = \frac{3}{137N} \quad , \quad N = 1,\ldots,5$$

Now because of the deductible of 2, the net annual premium $P = E[X]$ where

$$X = \begin{cases} 0 & , \text{ if } \quad N \le 2 \\ N-2 & , \text{ if } \quad N > 2 \end{cases}$$

Then,

$$P = E[X] = \sum_{N=3}^{5}(N-2)\frac{3}{137N} = (1)\left(\frac{1}{137}\right) + 2\left[\frac{3}{137(4)}\right] + 3\left[\frac{3}{137(5)}\right] = 0.0314$$

6. E

The line $5y - 4x = 3$ has slope $4/5$. It follows that we need to find t such that

$$\frac{4}{5} = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2t}{2t+1}$$

$$8t + 4 = 10t$$

$$2t = 4$$

$$t = 2$$

7. A

$$\begin{aligned}\operatorname{Cov}(C_1, C_2) &= \operatorname{Cov}(X+Y,\ \ X+1.2Y)\\
&= \operatorname{Cov}(X, X) + \operatorname{Cov}(Y, X) + \operatorname{Cov}(X, 1.2Y) + \operatorname{Cov}(\mathrm{Y}, 1.2\mathrm{Y})\\
&= \operatorname{Var} X + \operatorname{Cov}(X, Y) + 1.2\operatorname{Cov}(X, Y) + 1.2\operatorname{Var} Y\\
&= \operatorname{Var} X + 2.2\operatorname{Cov}(X, Y) + 1.2\operatorname{Var} Y\end{aligned}$$

$$\operatorname{Var} X = E(X^2) - (E(X))^2 = 27.4 - 5^2 = 2.4$$

$$\operatorname{Var} Y = E(Y^2) - (E(Y))^2 = 51.4 - 7^2 = 2.4$$

$$\operatorname{Var}(X+Y) = \operatorname{Var} X + \operatorname{Var} Y + 2\operatorname{Cov}(X, Y)$$

$$\begin{aligned}\operatorname{Cov}(X, Y) &= \frac{1}{2}\left(\operatorname{Var}(X+Y) - \operatorname{Var} X - \operatorname{Var} Y\right)\\
&= \frac{1}{2}(8 - 2.4 - 2.4) = 1.6\end{aligned}$$

$$\operatorname{Cov}(C_1, C_2) = 2.4 + 2.2(1.6) + 1.2(2.4) = 8.8$$

7. Alternate solution:

We are given the following information:

$$C_1 = X + Y$$
$$C_2 = X + 1.2Y$$
$$E[X] = 5$$
$$E\left[X^2\right] = 27.4$$
$$E[Y] = 7$$
$$E\left[Y^2\right] = 51.4$$
$$\text{Var}[X + Y] = 8$$

Now we want to calculate

$$\begin{aligned}
\text{Cov}(C_1, C_2) &= \text{Cov}(X + Y, X + 1.2Y) \\
&= E\left[(X + Y)(X + 1.2Y)\right] - E[X + Y] \bullet E[X + 1.2Y] \\
&= E\left[X^2 + 2.2XY + 1.2Y^2\right] - \left(E[X] + E[Y]\right)\left(E[X] + 1.2E[Y]\right) \\
&= E\left[X^2\right] + 2.2E[XY] + 1.2E\left[Y^2\right] - (5+7)\left(5+(1.2)7\right) \\
&= 27.4 + 2.2E[XY] + 1.2(51.4) - (12)(13.4) \\
&= 2.2E[XY] - 71.72
\end{aligned}$$

Therefore, we need to calculate $E[XY]$ first. To this end, observe

$$\begin{aligned}
8 = \text{Var}[X + Y] = E\left[(X + Y)^2\right] - \left(E[X + Y]\right)^2 \\
&= E\left[X^2 + 2XY + Y^2\right] - \left(E[X] + E[Y]\right)^2 \\
&= E\left[X^2\right] + 2E[XY] + E\left[Y^2\right] - (5+7)^2 \\
&= 27.4 + 2E[XY] + 51.4 - 144 \\
&= 2E[XY] - 65.2
\end{aligned}$$

$$E[XY] = (8 + 65.2)/2 = 36.6$$

Finally,

$$\text{Cov}(C_{1,} C_2) = 2.2(36.6) - 71.72 = 8.8$$

8. D

The function $F(t)$, $1 \le t \le 7$, achieves its minimum value at one of the endpoints of the interval $1 \le t \le 7$ or at t such that

$$0 = F'(t) = e^{-t} - te^{-t} = e^{-t}(1-t)$$

Since $F'(t) < 0$ for $t > 1$, we see that $F(t)$ is a decreasing function on the interval $1 < t \le 7$. We conclude that

$$F(7) = 7e^{-7} = 0.0064$$

is the minimum value of F.

9. D

Let

C = event that patient visits a chiropractor
T = event that patient visits a physical therapist

We are given that

$$\Pr[C] = \Pr[T] + 0.14$$

$$\Pr(C \cap T) = 0.22$$

$$\Pr(C^c \cap T^c) = 0.12$$

Therefore,

$$0.88 = 1 - \Pr\left[C^c \cap T^c\right] = \Pr[C \cup T] = \Pr[C] + \Pr[T] - \Pr[C \cap T]$$

$$= \Pr[T] + 0.14 + \Pr[T] - 0.22$$

$$= 2\Pr[T] - 0.08$$

or

$$\Pr[T] = (0.88 + 0.08)/2 = 0.48$$

10. E

Note that the terms of $\sum_{n=1}^{\infty}\left(a_n+\frac{1}{n}\right)$ exceed the terms of the harmonic series for $a_n=1$, $a_n=\frac{1}{n}$, or $a_n=\frac{1}{n^2}$ and will thus fail to converge.

Moreover, the terms of the series

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}+\frac{1}{n}\right]=\sum_{n=1}^{\infty}\left(\frac{1}{2n}+\frac{1}{2n}\right)=\sum_{n=1}^{\infty}\frac{1}{n}$$

are seen to be identical to the terms of the divergent harmonic series. By contrast, the series

$$\sum_{n=1}^{\infty}\left[a_n+\frac{1}{n}\right]=\sum_{n=1}^{\infty}\left[\frac{1-n}{n^2}+\frac{1}{n}\right]=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

is seen to converge by the integral test, since

$$\int_1^{\infty}x^{-2}dx=-\frac{1}{x}\Big|_1^{\infty}=1<\infty$$

11. D

If a month with one or more accidents is regarded as success and k = the number of failures before the fourth success, then k follows a negative binomial distribution and the requested probability is

$$\Pr[k\geq 4]=1-\Pr[k\leq 3]=1-\sum_{k=0}^{3}\binom{3+k}{k}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^k$$

$$=1-\left(\frac{3}{5}\right)^4\left[\binom{3}{0}\left(\frac{2}{5}\right)^0+\binom{4}{1}\left(\frac{2}{5}\right)^1+\binom{5}{2}\left(\frac{2}{5}\right)^2+\binom{6}{3}\left(\frac{2}{5}\right)^3\right]$$

$$=1-\left(\frac{3}{5}\right)^4\left[1+\frac{8}{5}+\frac{8}{5}+\frac{32}{25}\right]$$

$$=0.2898$$

Alternatively the solution is

$$\left(\frac{2}{5}\right)^4+\binom{4}{1}\left(\frac{2}{5}\right)^4\frac{3}{5}+\binom{5}{2}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2+\binom{6}{3}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^3=0.2898$$

which can be derived directly or by regarding the problem as a negative binomial distribution with

i) success taken as a month with no accidents
ii) k = the number of failures before the fourth success, and
iii) calculating $\Pr[k\leq 3]$

12. E

Observe that

$$f'(x) = \int_0^x f''(t)\,dt$$

$$= \text{(area bounded by } f'' \text{ above the } x\text{-axis from 0 to } x)$$
$$- \text{(area bounded by } f'' \text{ below the } x\text{-axis from 0 to } x)$$

For $x \in [0,5]$, f'' is drawn such that the area bounded by f'' above the *x*-axis from 0 to *x* is strictly greater than the area bounded by f'' below the *x*-axis from 0 to *x*. Therefore, $f'(x) > 0$ for $0 < x \le 5$. We conclude that *f* is an increasing function on the interval $[0,5]$, and its maximum value occurs at $x = 5$.

13. E

$$F(y) = \Pr[Y \le y] = \Pr\left[10X^{0.8} \le y\right] = \Pr\left[X \le \left(\tfrac{Y}{10}\right)^{10/8}\right] = 1 - e^{-\left(Y/10\right)^{10/8}}$$

Therefore, $f(y) = F'(y) = \dfrac{1}{8}\left(\dfrac{Y}{10}\right)^{1/4} e^{-(Y/10)^{5/4}}$

14. A

L'Hôspital's Rule may be applied since *f* and *g* are given differentiable and from the diagram $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

Therefore $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{f'(x)}{g'(x)}$

Also from the diagram $\lim_{x\to 0} f'(x) = f'(0) > 0$

and $\lim_{x\to 0} g'(x) = g'(0) < 0$

This leads to $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{f'(x)}{g'(x)} = \dfrac{f'(0)}{g'(0)} < 0$

15. E

Let $X_1,\ldots,X_{100}$ denote the number of pensions that will be provided to each new recruit. Now under the assumptions given,

$$X_i = \begin{cases} 0 & \text{with probability} \quad 1-0.4=0.6 \\ 1 & \text{with probability} \quad (0.4)(0.25)=0.1 \\ 2 & \text{with probability} \quad (0.4)(0.75)=0.3 \end{cases}$$

for $i=1,\ldots,100$. Therefore,

$$E[X_i]=(0)(0.6)+(1)(0.1)+(2)(0.3)=0.7 \text{ ,}$$

$$E\left[X_i^{\,2}\right]=(0)^2(0.6)+(1)^2(0.1)+(2)^2(0.3)=1.3 \text{ , and}$$

$$\operatorname{Var}[X_i]=E\left[X_i^{\,2}\right]-\{E[X_i]\}^2=1.3-(0.7)^2=0.81$$

Since $X_1,\ldots,X_{100}$ are assumed by the consulting actuary to be independent, the Central Limit Theorem then implies that $S=X_1+\ldots+X_{100}$ is approximately normally distributed with mean

$$E[S]=E[X_1]+\ldots+E[X_{100}]=100(0.7)=70$$

and variance

$$\operatorname{Var}[S]=\operatorname{Var}[X_1]+\ldots+\operatorname{Var}[X_{100}]=100(0.81)=81$$

Consequently,

$$\begin{aligned}\Pr[S\le 90.5]&=\Pr\left[\frac{S-70}{9}\le\frac{90.5-70}{9}\right]\\&=\Pr[Z\le 2.28]\\&=0.99\end{aligned}$$

16. C

Observe

$$\begin{aligned}\Pr[4<S<8]&=\Pr\left[4<S<8\,\middle|\,N=1\right]\Pr[N=1]+\Pr\left[4<S<8\,\middle|\,N>1\right]\Pr[N>1]\\&=\frac{1}{3}\left(e^{-4/5}-e^{-8/5}\right)+\frac{1}{6}\left(e^{-1/2}-e^{-1}\right)\quad *\\&=0.122\end{aligned}$$

*Uses that if X has an exponential distribution with mean μ

$$\Pr(a\le X\le b)=\Pr(X\ge a)-\Pr(X\ge b)=\int_a^\infty\frac{1}{\mu}e^{-t/\mu}dt-\int_b^\infty\frac{1}{\mu}e^{-t/\mu}dt=e^{-\frac{a}{\mu}}-e^{-\frac{b}{\mu}}$$

17. B

Note that

$$\begin{aligned}\Pr[X > x] &= \int_x^{20} 0.005(20-t)\,dt \\ &= 0.005\left(20t - \frac{1}{2}t^2\right)\Big|_x^{20} \\ &= 0.005\left(400 - 200 - 20x + \frac{1}{2}x^2\right) \\ &= 0.005\left(200 - 20x + \frac{1}{2}x^2\right)\end{aligned}$$

where $0 < x < 20$. Therefore,

$$\Pr\left[X > 16 \middle| X > 8\right] = \frac{\Pr[X > 16]}{\Pr[X > 8]} = \frac{200 - 20(16) + \tfrac{1}{2}(16)^2}{200 - 20(8) + \tfrac{1}{2}(8)^2} = \frac{8}{72} = \frac{1}{9}$$

18. D

Note that

$$f'(x) = \begin{cases} 2x & \text{for} \quad x < 0 \quad \text{and} \quad x > 1 \\ \dfrac{1}{2\sqrt{x}} & \text{for} \quad 0 < x < 1 \end{cases}$$

Furthermore,

$$0 = \lim_{x \to 0^-} f'(x) \neq \lim_{x \to 0^+} f'(x) = +\infty$$

and

$$2 = \lim_{x \to 1^+} f'(x) \neq \lim_{x \to 1^-} f'(x) = \frac{1}{2}$$

so f is not differentiable at $x = 0$ or $x = 1$, although f is differentiable everywhere else. By contrast, f is continuous everywhere since

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1$$

19. B

The amount of money the insurance company will have to pay is defined by the random variable

$$Y = \begin{cases} 1000x & \text{if} \quad x < 2 \\ 2000 & \text{if} \quad x \ge 2 \end{cases}$$

where x is a Poisson random variable with mean 0.6 . The probability function for X is

$$p(x) = \frac{e^{-0.6}(0.6)^k}{k!} \qquad k = 0,1,2,3\cdots \text{ and}$$

$$E[Y] = 0 + 1000(0.6)e^{-0.6} + 2000e^{-0.6}\sum_{k=2}^{\infty}\frac{0.6^k}{k!}$$

$$= 1000(0.6)e^{-0.6} + 2000\left(e^{-0.6}\sum_{k=0}^{\infty}\frac{0.6^k}{k!} - e^{-0.6} - (0.6)e^{-0.6}\right)$$

$$= 2000e^{-0.6}\sum_{k=0}^{\infty}\frac{(0.6)^k}{k!} - 2000e^{-0.6} - 1000(0.6)e^{-0.6} = 2000 - 2000e^{-0.6} - 600e^{-0.6}$$

$$= 573$$

$$E\left[Y^2\right] = (1000)^2(0.6)e^{-0.6} + (2000)^2 e^{-0.6}\sum_{k=2}^{\infty}\frac{0.6^k}{k!}$$

$$= (2000)^2 e^{-0.6}\sum_{k=0}^{\infty}\frac{0.6^k}{k!} - (2000)^2 e^{-0.6} - \left[(2000)^2 - (1000)^2\right](0.6)e^{-0.6}$$

$$= (2000)^2 - (2000)^2 e^{-0.6} - \left[(2000)^2 - (1000)^2\right](0.6)e^{-0.6}$$

$$= 816,893$$

$$\text{Var}[Y] = E\left[Y^2\right] - \left\{E[Y]\right\}^2$$

$$= 816,893 - (573)^2$$

$$= 488,564$$

$$\sqrt{\text{Var}[Y]} = 699$$

20. B

The graph of E' tells us that employment increases from April through August (because $E'(t) > 0$ then) and does not increase from August through April (since $E'(t) \le 0$ then). It follows that employment is a minimum in April.

21. D

Let N_1 and N_2 denote the number of claims during weeks one and two, respectively. Then since N_1 and N_2 are independent,

$$\begin{aligned}\Pr[N_1+N_2=7] &= \sum_{n=0}^{7}\Pr[N_1=n]\Pr[N_2=7-n] \\ &= \sum_{n=0}^{7}\left(\frac{1}{2^{n+1}}\right)\left(\frac{1}{2^{8-n}}\right) \\ &= \sum_{n=0}^{7}\frac{1}{2^9} \\ &= \frac{8}{2^9}=\frac{1}{2^6}=\frac{1}{64}\end{aligned}$$

22. D

The amount of the drug present peaks at the instants an injection is administered. If $A(n)$ is the amount of the drug remaining in the patient's body at the time of the nth injection, then

$$\begin{aligned}A(1) &= 250 \\ A(2) &= 250\left(1+e^{-1/6}\right) \\ &\vdots \\ A(n) &= 250\sum_{k=0}^{n-1}e^{-k/6}\end{aligned}$$

and the least upper bound is

$\lim\limits_{n\to\infty}A(n)=250\sum_{k=0}^{\infty}e^{-k/6}=\dfrac{250}{1-e^{-1/6}}=1628$ (noting the series is geometric with ratio $e^{-1/6}$)

23. C

Observe that

$$\sin(\theta+\pi)+\sqrt{3}\cos(\theta+\pi) = -\left[\sin\theta+\sqrt{3}\cos\theta\right]$$

Therefore, (r,θ) and $(-r,\theta+\pi)$ define the same point, and we may restrict attention to θ such that $0 \le \theta < \pi$. Now for $0 \le \theta \le \frac{\pi}{2}$, $r > 0$ and the graph of *r* is in the first quadrant (i.e., to the right of $\theta = \frac{\pi}{2}$). On the other hand, for $\frac{\pi}{2} < \theta < \pi$, $r > 0$ when

$$\sin\theta+\sqrt{3}\cos\theta > 0$$
$$\sin\theta > \sqrt{3}\cos\theta$$
$$\frac{\sin\theta}{\cos\theta} > \sqrt{3}$$
$$\tan\theta > \sqrt{3}$$

This is true in the interval $\left[\frac{\pi}{2}, \pi\right]$ when $\frac{\pi}{2} < \theta < \frac{2\pi}{3}$. Consequently, the area bounded by the graph of *r* to the left of $\theta = \frac{\pi}{2}$ is given by $\int_{\pi/2}^{2\pi/3} r^2 d\theta$.

[It is not needed to do the problem, but the graph of the curve is a circle of radius 1 centered at $x = \frac{\sqrt{3}}{2}$, $y = \frac{1}{2}$].

24. D

$100x^{1/4}y^{1/2}$ must be maximized subject to $x + y = 150,000$

Since $y = 150,000 - x$, this reduces to maximizing

$S(x) = 100x^{1/4}(150,000 - x)^{1/2}$, $0 \le x \le 150,000$

$S'(x) = 25x^{-3/4}(150,000 - x)^{1/2} - 50x^{1/4}(150,000 - x)^{-1/2} = 0$

$(150,000 - x) - 2x = 0$

$3x = 150,000$

$x = \ \ 50,000$

(This value of x is a maximum since $S'(x) > 0$ for $0 < x < 50,000$ and $S'(x) < 0$ for $50,000 < x < 150,000$).

Maximum sales are then $100(50,000)^{1/4}(150,000 - 50,000)^{1/2} = 472,871$

Alternate solution using Lagrange Multipliers

Solve $\quad x + y - 150,000 = 0$

$$\frac{\partial}{\partial x}100x^{1/4}y^{1/2} = \lambda \frac{\partial}{\partial x}(x + y - 150,000)$$

$$\frac{\partial}{\partial y}100x^{1/4}y^{1/2} = \lambda \frac{\partial}{\partial y}(x + y - 150,000)$$

From the last two equations

$$25x^{-3/4}y^{1/2} = \lambda$$

$$50x^{1/4}y^{-1/2} = \lambda$$

Eliminating λ

$$25x^{-3/4}y^{1/2} = 50x^{1/4}y^{-1/2}$$

$$25y = 50x$$

$$y = 2x$$

Using the first equation

$$x + 2x - 150,000 = 0$$

$$x = 50,000$$

$$y = 100,000$$

The extreme value (which must be a maximum) is $100(50,000)^{1/4}(100,000)^{1/2} = 472,871$

25. E

$$\Pr\left[1 < Y < 3 \middle| X = 2\right] = \int_1^3 \frac{f(2,y)}{f_X(2)} dy$$

$$f(2,y) = \frac{2}{4(2-1)} y^{-(4-1)/2-1} = \frac{1}{2} y^{-3}$$

$$f_X(2) = \int_1^\infty \frac{1}{2} y^{-3} dy = -\frac{1}{4} y^{-2} \Big|_1^\infty = \frac{1}{4}$$

Finally, $\Pr\left[1 < Y < 3 \middle| X = 2\right] = \dfrac{\int_1^3 \frac{1}{2} y^{-3} dy}{\frac{1}{4}} = -y^{-2}\Big|_1^3 = 1 - \dfrac{1}{9} = \dfrac{8}{9}$

26. C

The increase in sales from year two to year four that is attributable to the advertising campaign is given by

$$\int_2^4 \left[\left(t^2 + \frac{1}{2}\right) - \left(t + \frac{5}{2}\right)\right] dt = \int_2^4 \left(t^2 - t - 2\right) dt$$

$$= \left(\frac{t^3}{3} - \frac{t^2}{2} - 2t\right)\Big|_2^4$$

$$= \frac{64}{3} - 8 - 8 - \frac{8}{3} + 2 + 4$$

$$= \frac{56}{3} - 10 = \frac{26}{3}$$

27. D

Let X denote the number of employees that achieve the high performance level. Then X follows a binomial distribution with parameters $n = 20$ and $p = 0.02$. Now we want to determine x such that

$$\Pr[X > x] \le 0.01$$

or, equivalently,

$$0.99 \le \Pr[X \le x] = \sum_{k=0}^{x} \binom{20}{k} (0.02)^k (0.98)^{20-k}$$

The following table summarizes the selection process for x:

x	$\Pr[X = x]$	$\Pr[X \le x]$
0	$(0.98)^{20} = 0.668$	0.668
1	$20(0.02)(0.98)^{19} = 0.272$	0.940
2	$190(0.02)^2(0.98)^{18} = 0.053$	0.993

Consequently, there is less than a 1% chance that more than two employees will achieve the high performance level. We conclude that we should choose the payment amount C such that

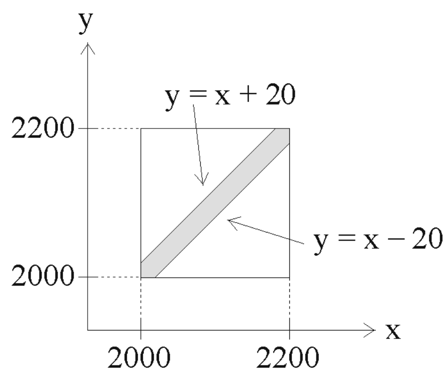
$$2C = 120,000$$

or

$$C = 60,000$$

28. B

Let X and Y denote the two bids. Then the graph below illustrates the region over which X and Y differ by less than 20:



Based on the graph and the uniform distribution:

$$\Pr\left[\left|X-Y\right|<20\right]=\frac{\text{Shaded Region Area}}{(2200-2000)^2}=\frac{200^2-2\cdot\frac{1}{2}(180)^2}{200^2}$$

$$=1-\frac{180^2}{200^2}=1-(0.9)^2=0.19$$

More formally (still using symmetry)

$$\Pr\left[\left|X-Y\right|<20\right]=1-\Pr\left[\left|X-Y\right|\geq 20\right]=1-2\Pr\left[X-Y\geq 20\right]$$

$$=1-2\int_{2020}^{2200}\int_{2000}^{x-20}\frac{1}{200^2}\,dydx=1-2\int_{2020}^{2200}\frac{1}{200^2}\,y\Big|_{2000}^{x-20}dx$$

$$=1-\frac{2}{200^2}\int_{2020}^{2200}(x-20-2000)\,dx=1-\frac{1}{200^2}(x-2020)^2\Big|_{2020}^{2200}$$

$$=1-\left(\frac{180}{200}\right)^2=0.19$$

29. B

Let X and Y denote repair cost and insurance payment, respectively, in the event the auto is damaged. Then

$$Y = \begin{cases} 0 & \text{if} \quad x \le 250 \\ x - 250 & \text{if} \quad x > 250 \end{cases}$$

and

$$E[Y] = \int_{250}^{1500} \frac{1}{1500}(x-250)dx = \frac{1}{3000}(x-250)^2 \Big|_{250}^{1500} = \frac{1250^2}{3000} = 521$$

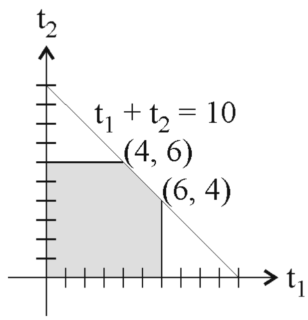
$$E\left[Y^2\right] = \int_{250}^{1500} \frac{1}{1500}(x-250)^2\, dx = \frac{1}{4500}(x-250)^3 \Big|_{250}^{1500} = \frac{1250^3}{4500} = 434,028$$

$$\text{Var}[Y] = E\left[Y^2\right] - \{E[Y]\}^2 = 434,028 - (521)^2$$

$$\sqrt{\text{Var}[Y]} = 403$$

30. C

Let $f(t_1, t_2)$ denote the joint density function of T_1 and T_2. The domain of *f* is pictured below:



Now the area of this domain is given by

$$A = 6^2 - \frac{1}{2}(6-4)^2 = 36 - 2 = 34$$

Consequently,

$$f(t_1, t_2) = \begin{cases} \dfrac{1}{34} & , \ 0 < t_1 < 6 \ , \ 0 < t_2 < 6 \ , \ t_1 + t_2 < 10 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\begin{aligned} E[T_1 + T_2] &= E[T_1] + E[T_2] = 2E[T_1] \qquad \text{(due to symmetry)} \\ &= 2\left\{ \int_0^4 t_1 \int_0^6 \frac{1}{34} dt_2 dt_1 + \int_4^6 t_1 \int_0^{10-t_1} \frac{1}{34} dt_2 dt_1 \right\} \\ &= 2\left\{ \int_0^4 t_1 \left[\frac{t_2}{34} \Big|_0^6 \right] dt_1 + \int_4^6 t_1 \left[\frac{t_2}{34} \Big|_0^{10-t_1} \right] dt_1 \right\} \\ &= 2\left\{ \int_0^4 \frac{3t_1}{17} dt_1 + \int_4^6 \frac{1}{34}\left(10t_1 - t_1^2\right) dt_1 \right\} \\ &= 2\left\{ \frac{3t_1^2}{34} \Bigg|_0^4 + \frac{1}{34}\left(5t_1^2 - \frac{1}{3}t_1^3\right) \Bigg|_4^6 \right\} \\ &= 2\left\{ \frac{24}{17} + \frac{1}{34}\left[180 - 72 - 80 + \frac{64}{3} \right] \right\} \\ &= 5.7 \end{aligned}$$

31. E

The general solution of the differential equation may be determined as follows:

$$\int \frac{1}{y}\,dy = \int (k_1 - k_2)\,dt$$

$$\ln(y) = (k_1 - k_2)t + C \qquad \text{(C is a constant)}$$

$$y(t) = e^c e^{(k_1 - k_2)t}$$

When $t = 0$,

$y(0) = e^c$, so

$$y(t) = y(0)e^{(k_1 - k_2)t}$$

Now we are given that if $k_1 = 0,\ y(8) = \frac{1}{2}y(0)$.

Therefore,

$$\frac{1}{2}y(0) = y(8) = y(0)e^{-8k_2}$$

$$\frac{1}{2} = e^{-8k_2}$$

$$8k_2 = \ln(2)$$

$$k_2 = \frac{1}{8}\ln(2)$$

[Note the problem also gives $y(24) = 2y(0) = y(0)e^{24(k_1 - k_2)}$, but that information is not needed to determine k_2].

32. B

Observe

$$\Pr\left[N \ge 1 \middle| N \le 4\right] = \frac{\Pr[1 \le N \le 4]}{\Pr[N \le 4]} = \left[\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}\right] \Big/ \left[\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}\right]$$

$$= \frac{10 + 5 + 3 + 2}{30 + 10 + 5 + 3 + 2} = \frac{20}{50} = \frac{2}{5}$$

33. E

Let X and Y denote the year the device fails and the benefit amount, respectively. Then the density function of X is given by

$$f(x) = (0.6)^{x-1}(0.4) \quad , \quad x = 1, 2, 3...$$

and

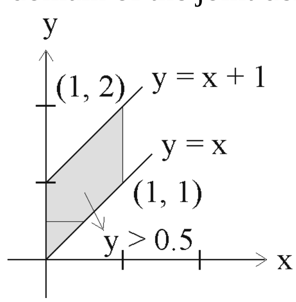
$$y = \begin{cases} 1000(5-x) & \text{if} \quad x = 1, 2, 3, 4 \\ 0 & \text{if} \quad x > 4 \end{cases}$$

It follows that

$$E[Y] = 4000(0.4) + 3000(0.6)(0.4) + 2000(0.6)^2(0.4) + 1000(0.6)^3(0.4)$$
$$= 2694$$

34. D

The diagram below illustrates the domain of the joint density $f(x, y)$ of X and Y.



We are told that the marginal density function of X is $f_x(x) = 1$, $0 < x < 1$

while $f_{y|x}(y|x) = 1$, $x < y < x+1$

It follows that $f(x, y) = f_x(x)\, f_{y|x}(y|x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \text{ , } x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$

Therefore,

$$\Pr[Y > 0.5] = 1 - \Pr[Y \le 0.5] = 1 - \int_0^{1/2}\int_x^{1/2} dy dx$$
$$= 1 - \int_0^{1/2} y\Big|_x^{1/2} dx = 1 - \int_0^{1/2}\left(\frac{1}{2} - x\right)dx$$
$$= 1 - \left[\frac{1}{2}x - \frac{1}{2}x^2\right]\Bigg|_0^{1/2} = 1 - \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

[Note since the density is constant over the shaded parallelogram in the figure the solution is also obtained as the ratio of the area of the portion of the parallelogram above $y = 0.5$ to the entire shaded area.]

35. E

If X is the random variable representing claim amounts, the probability that X exceeds the deductible is

$$\Pr[X > 1] = \int_1^\infty xe^{-x}dx = -xe^{-x}\Big|_1^\infty + \int_1^\infty e^{-x}dx \qquad \text{(integration by parts)}$$

$$= e^{-1} - e^{-x}\Big|_1^\infty = e^{-1} + e^{-1} = 2e^{-1}$$

$$= 0.736$$

It follows that the company expects $(100)(0.736) = 74$ claims.

36. C

Let

x = price in excess of 60 that the company charges,
$p(x)$ = price per policy that the company charges,
$n(x)$ = number of policies the company sells per month, and
$R(x)$ = revenue per month the company collects

Then

$$p(x) = 60 + x$$

$$n(x) = 80 - x$$

$$R(x) = n(x)\,p(x) = (80 - x)(60 + x)$$

$$R'(x) = -(60 + x) + (80 - x) = 20 - 2x$$

$$R''(x) = -2 < 0$$

It follows that $R(x)$ is a maximum when $R'(x) = 0$. We conclude that

$$20 - 2x = 0$$

$$x = 10$$

and $\quad R(10) = (80 - 10)(60 + 10) = 4900$

when $R(x)$ is a maximum.

37. A

The joint density of T_1 and T_2 is given by

$$f(t_1,t_2) = e^{-t_1}e^{-t_2} \quad , \quad t_1 > 0 \quad , \quad t_2 > 0$$

Therefore,

$$\Pr[X \le x] = \Pr[2T_1 + T_2 \le x]$$

$$= \int_0^x \int_0^{\frac{1}{2}(x-t_2)} e^{-t_1}e^{-t_2}\,dt_1 dt_2 = \int_0^x e^{-t_2}\left[-e^{-t_1}\Big|_0^{\frac{1}{2}(x-t_2)}\right]dt_2$$

$$= \int_0^x e^{-t_2}\left[1-e^{-\frac{1}{2}x+\frac{1}{2}t_2}\right]dt_2 = \int_0^x \left(e^{-t_2} - e^{-\frac{1}{2}x}e^{-\frac{1}{2}t_2}\right)dt_2$$

$$= \left[-e^{-t_2} + 2e^{-\frac{1}{2}x}e^{-\frac{1}{2}t_2}\right]\Big|_0^x = -e^{-x} + 2e^{-\frac{1}{2}x}e^{-\frac{1}{2}x} + 1 - 2e^{-\frac{1}{2}x}$$

$$= 1 - e^{-x} + 2e^{-x} - 2e^{-\frac{1}{2}x} = 1 - 2e^{-\frac{1}{2}x} + e^{-x} \quad , \quad x > 0$$

It follows that the density of X is given by

$$g(x) = \frac{d}{dx}\left[1 - 2e^{-\frac{1}{2}x} + e^{-x}\right]$$

$$= e^{-\frac{1}{2}x} - e^{-x} \quad , \quad x > 0$$

38. E

Let $X_1, X_2,$ and X_3 denote annual loss due to storm, fire, and theft, respectively. In addition, let $Y = Max(X_1, X_2, X_3)$.

Then

$$\Pr[Y > 3] = 1 - \Pr[Y \le 3] = 1 - \Pr[X_1 \le 3]\Pr[X_2 \le 3]\Pr[X_3 \le 3]$$

$$= 1 - \left(1-e^{-3}\right)\left(1-e^{-3/1.5}\right)\left(1-e^{-3/2.4}\right) \qquad *$$

$$= 1 - \left(1-e^{-3}\right)\left(1-e^{-2}\right)\left(1-e^{-5/4}\right)$$

$$= 0.414$$

* Uses that if X has an exponential distribution with mean μ

$$\Pr(X \le x) = 1 - \Pr(X \ge x) = 1 - \int_x^{\infty} \frac{1}{\mu}e^{-t/\mu}dt = 1 - \left(-e^{-t/\mu}\right)\Big|_x^{\infty} = 1 - e^{-x/\mu}$$

39. D

We are given that

$$P(x) = -x^2 + 50x - 25$$

before the translation occurs. The revised profit function $P^*(x)$ may be determined as follows:

$$\begin{aligned} P^*(x) &= P(x-2) - 3 \\ &= -(x-2)^2 + 50(x-2) - 25 - 3 \\ &= -x^2 + 4x - 4 + 50x - 100 - 28 \\ &= -x^2 + 54x - 132 \end{aligned}$$

40. B

Observe that

$$E[X+Y] = E[X] + E[Y] = 50 + 20 = 70$$

$$Var[X+Y] = Var[X] + Var[Y] + 2\ Cov[X,Y] = 50 + 30 + 20 = 100$$

for a randomly selected person. It then follows from the Central Limit Theorem that T is approximately normal with mean

$$E[T] = 100(70) = 7000$$

and variance

$$Var[T] = 100(100) = 100^2$$

Therefore,

$$\begin{aligned} \Pr[T < 7100] &= \Pr\left[\frac{T-7000}{100} < \frac{7100-7000}{100}\right] \\ &= \Pr[Z < 1] = 0.8413 \end{aligned}$$

where Z is a standard normal random variable.